DELTA METHOD

for solving quadratic equations
and special cubic equations

Frederick Herrmann

Delta Method

for solving quadratic equations and special cubic equations

Frederick Neil Herrmann

Makua Lani Christian Academy
Kailua-Kona, Hawaii

June 2024

INTRODUCTION

This method for solving quadratic roots has a singular benefit: It is easy to understand what the method is doing. Therefore, it is beneficial for teachers and students. I refer to this method as the Delta Method because it measures the amount of change from the maximum area. As such, it can be taught in conjunction with maximum and minimum values. The method works with monic quadratics; any other quadratic must be divided by a. The method can be applied to cubic equations in which, given three real roots, two of the roots are equidistant from the center root.

THE DELTA METHOD for QUADRATIC EQUATIONS

Consider the construction of a fence, such that the length and width of the fence are given by l and w. Let these dimensions be placed in two factors given as $(x + l)$ and $(x + w)$. Constructing a parabola from their product, we have:

$$y = x^2 + (l + w)x + lw$$

We see that the middle term is half the perimeter of the fence and that the third term is the area. Now if you are a high school mathematics teacher, you are probably familiar with the fact that the shape that provides the greatest area per perimeter length is the square, that is, where length and width are equal. Let's call this length s and view the equation.

$$y = x^2 + 2sx + s^2$$

We will now analyze how much the area changes when we keep the half-perimeter constant, but change the value of length and width incrementally away from s. The symbol δ will represent the increment and the two factors will be expressed as$(x + s - \delta)$ and $(x + s + \delta)$. The symbol D will represent the change in area from the original square. Here is an example:

Factors	δ	Equation	*D*
$(x + 13)(x + 13)$	0	$y = x^2 + 26x + 169$	0
$(x + 12)(x + 14)$	1	$y = x^2 + 26x + 168$	1
$(x + 11)(x + 15)$	2	$y = x^2 + 26x + 165$	4
$(x + 10)(x + 16)$	3	$y = x^2 + 26x + 160$	9
$(x + 9)(x + 17)$	4	$y = x^2 + 26x + 153$	16

Clearly, D is simply δ^2. Algebraically, we can see this here; it works something like a conjugate.

$$y = (x + s - \delta)(x + s + \delta)$$

$$y = x^2 + 2sx + \left(s^2 - \delta^2\right)$$

We are now prepared to develop a method for finding the roots. Once we know the value of D, we can find the value of δ. So here we go!

EXAMPLE 1

Our equation is

$$y = x^2 + 12x + 10$$

The value of s is 6 and therefore the perfect square is 36. The difference of the area from the square 26. Therefore, δ is the square root of 26. So our equation is

$$y = \left(x + 6 - \sqrt{26}\right)\left(x + 6 + \sqrt{26}\right)$$

Remember these factors equal zero, so the sign must change when we solve for x to find the roots. The roots are

$$R = \ -6 - \sqrt{26},\ \ -6 + \sqrt{26}$$

EXAMPLE 2

Our equation is

$$y = x^2 + 12x - 10$$

The value of s is 6 and therefore the perfect square is 36. The difference of the area from the square 46. Therefore, δ is the square root of 46. So our equation is

$$y = \left(x + 6 - \sqrt{46}\right)\left(x + 6 + \sqrt{46}\right)$$

These factors equal zero, so we need to change the signs when we solve for x. The roots are

$$R = \quad -6 - \sqrt{46}, \quad -6 + \sqrt{46}$$

STUDENT COGNITIVE PROCESSING

Sample problem: $y = x^2 + 8x + 10$	
What is the length of the side of the perfect square?	Student divides b by 2. The result is 4.
What would the maximum area be if that were the side length?	The student squares 4 and arrives at 16.
What is the difference between this area and the given area?	The students subtracts 10 from 16. The result is 6.
What is the difference in length and width from the side of the square?	The student takes the square root of 6. Since this number does not come out "clean," the student leaves the result under the radical: $\sqrt{6}$
The student writes down the factors. (A student may likely skip this step.)	$y = \left(x + 4 - \sqrt{6}\right)\left(x + 4 + \sqrt{6}\right)$
The student writes down the roots.	$x = \quad -4 - \sqrt{6}, \quad -4 + \sqrt{6}$

Sample problem: $y = x^2 + 5x - 6$	
What is the length of the side of the perfect square?	Student divides b by 2. The result is 2.5.
What would the maximum area be if that were the side length?	The student squares 2.5 and arrives at 6.25.
What is the difference between this area and the given area?	The students subtracts -6 from 6.25. The result is 12.25.
What is the difference in length and width from the side of the square?	The student takes the square root of 12.25. The result is 3.5.
The student writes down the factors.	$y = (x + 2.5 - 3.5)(x + 2.5 + 3.5)$ $y = (x - 1)(x + 6)$
The student writes down the roots.	$x = 1, -6$

THE DELTA METHOD for CUBIC EQUATIONS

We can use the same logic for a cubic equation on the condition that the cubic center has three real roots with two of the roots equidistant from the center root. In such a case, the equation could be seen as the product of three factors as seen here.

$$y = (x + s - \delta)(x + s)(x + s + \delta)$$

When we multiply these factors, the equation looks like

$$y = x^3 + 3sx^2 + \left(3s^2 - \delta^2\right)x + s\left(s^2 - \delta^2\right)$$

We can find s from the second term and solve for δ using either the third or fourth term. Let's try!

EXAMPLE

Our equation is

$$y = x^3 + 15x^2 + 66x + 80$$

The value of s is 5. Now we have using the equation of the third term:

$$3s^2 - \delta^2 = 66$$

$$3 \cdot 5^2 - \delta^2 = 66$$

Solving:

$$75 - \delta^2 = 66$$

$$\delta^2 = 9$$

$$\delta = 3$$

Therefore, we have as our factors

$$y = (x + s - \delta)(x + s)(x + s + \delta)$$

$$y = (x + 5 - 3)(x + 5)(x + 5 + 3)$$

$$y = (x + 2)(x + 5)(x + 8)$$

So the roots are $-2, \ -5, \ -8$.

Let's double-check using the equation from the fourth term. There we have

$$s\left(s^2 - \delta^2\right) = 80$$

$$5\left(5^2 - \delta^2\right) = 80$$

$$25 - \delta^2 = 16$$

$$\delta^2 = 9$$

$$\delta = 3$$

This confirms the previous result.

POSTSCRIPT

As usual, I apologize if anyone has developed this method before me. Amateurs like myself can be quite ignorant of everything that is out there. When put into algebraic form as an equation, this method (at least for the quadratic) is very similar to the equation put forward by Po-Shen Loh (2019). However, its development and logic are completely different, thus the method is uniquely performed.

I have previously published a much longer work entitled *Ten Novel Methods for Solving Quadratic Roots* (2023). After teaching the Vertex Method described therein for two years now, I can attest that it is very popular with my students and they prefer it over the Quadratic Formula.

Thank you to Christian Williams for proofreading and verifying this paper. Mahalo!