The Hang Time Method

for solving projectile height and velocity

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PREFACE

Here is my usual preface where I declare that this method and several of these equations are original to the best of my knowledge, and, if not, they are at least "original to me" and I apologize if anyone else has gotten here first.

In fact, I am a bit astonished that I have not been able to find this method elsewhere, whether online or on paper, since the mathematics is very straightforward and easily derived. But my own discovery was by necessity as I was intentionally looking for a mathematically simple lab that I could use with young students knowing only basic algebra.

Thank you to Christian Williams for verifying this work.

THE EQUATIONS FOR GROUND LAUNCH

Having been asked to teach an 8th grade physics class, I was looking for a classical mechanics lab—easy on mathematics—for projectile motion. Trigonometry was not an option: The lab needed to rely only on algebra.

My searches for works that taught students how to derive "initial vertical velocity" were somewhat complicated: Those without trigonometry required at least two known values, and I could not apply these methods with my students while performing a lab on a ball field. For example, we could not know "as givens" the initial velocity of a ball as a student threw it upwards, nor the velocity when it hit the ground, nor (without applying trigonometry) its maximum height. We could only know, using a stopwatch, when it left the student's hand and when it hit the ground.

But it occurred to me that this would be enough. I remembered something I had written in *Mathematica Exploratio I* about projectile motion, namely the model rocket problem:

$$time\ of\ rocket\ impact\ =\ -\frac{initial\ velocity}{gravity}+\frac{impact\ velocity}{gravity}$$

$$time\ of\ rocket\ impact\ =\ time\ to\ climb\ +\ time\ to\ fall$$

These are both forms of the Quadratic Formula. For the rocket problem, the time of impact is always positive, so the terms are always added. Of course, this would apply not just to rockets but to anything thrown up in the air.

What we want is something easy, so let us begin by assuming that there is no rocket platform; in other words, we let $c = 0$ in the equation,

$$y = ax^2 + bx + c$$

However, for our purposes, it is best to use the equation in the form

$$H(t) = -5t^2 + v_0 t + h_0$$

where v_0 is the initial velocity and h_0 is the initial or platform height. So the simplest method to begin with is to allow $h_0 = 0$.

Let us look again at the Quadratic Formula for the model rocket.

$$t_h = -\frac{v_0}{gravity} + \frac{v_f}{gravity}$$

We are using t_h for hang time and v_f for final velocity. When our model rocket launches from the ground, its initial velocity and final velocity are the same. So we can adjust and solve the equation accordingly.

$$t_h = -\frac{v_0}{g} + \frac{v_0}{g} = \frac{2v_0}{g}$$

$$v_0 = \frac{g}{2} \cdot t_h$$

I ignore the negative, though it could certainly apply to final velocity. With my students, I use $g = 10\ m/s^2$ to simplify the math. So our initial equation becomes

$$v_0 = -5 \cdot t_h\ [m/s]$$

This is a very simple equation that is suited to middle school students. Using a stopwatch, the students can measure the hang time and use that data point to discover the initial velocity. A fastidious teacher will point out to the students that, for the data to be very precise, they really want to stop the stopwatch at the height from which is launched, which will most likely be the height of the hand or foot of the student launching the projectile—since the projectile is not actually launching from the ground.

Students can take the lab one further step by finding the projectile's maximum height. A special characteristic of the launch-from-the-ground parabola is that the projectile is at maximum height at half the hang time. Students can build the equation using the initial velocity that they calculated from the hang time. The equation will look like this:

$$H(t) = -5t^2 + v_0 t$$

To find maximum height, the students need to cut the hang time in half and plug it into the equation. Essentially,

$$H_{max} = -5\left(0.5t_h\right)^2 + v_0\left(0.5t_h\right)$$

My students were able to solve for both initial velocity and maximum height using these equations. And they had a little bit of fun with this, since they competed at how high they could get the projectiles to go. For maximum height, it seemed the volleyball worked the best. For less competitive students, I had Nerf guns available.

THE EQUATIONS FOR PLATFORM LAUNCH

For teachers that want to be more precise or want to challenge students further, an equation can be used for initial velocity that takes into account the height from which the projectile actually launches. This is analogous to the model rocket launching from a platform. The derivation of the equation takes a little time. For teachers accustomed to the Quadratic Formula in its usual form, simply keep in mind the following:

$$a = \frac{g}{2}$$

$$b = v_0$$

$$c = h_0$$

Here is the derivation:

$$t_h = -\frac{v_0}{g} + \frac{v_f}{g}$$

$$t_h \cdot g = -v_0 + v_f$$

$$t_h \cdot g = -v_0 + \sqrt{v_0^2 - 4\frac{g}{2}h_0} = -v_0 + \sqrt{v_0^2 - 2gh_0}$$

$$\sqrt{v_0^2 - 2gh_0} = t_h \cdot g + v_0$$

$$v_0^2 - 2gh_0 = t_h^2 \cdot g^2 + v_0^2 - 2v_0 t_h g$$

$$-2gh_0 = t_h^2 \cdot g^2 - 2v_0 t_h g$$

$$2v_0 t_h g = 2gh_0 + t_h^2 \cdot g^2$$

$$2v_0 t_h = 2h_0 + t_h^2 \cdot g$$

$$v_0 t_h = h_0 + t_h^2 \cdot \frac{g}{2}$$

$$v_0 = \frac{h_0}{t_h} + t_h \cdot \frac{g}{2}$$

Note that the second term is negative because $g = -5$. The result will be negative, so it is convenient to alter it to

$$v_0 = -\frac{h_0}{t_h} - t_h \cdot \frac{g}{2}$$

to demonstrate the actual launch velocity, rather than the falling velocity at the same height.

The students can build the equation. They will have

$$H(t) = -5t^2 + v_0 t + h_0$$

Can they solve for maximum height? Yes, if they use the well-known formula

$$t_{max} = \frac{-v_0}{g} = \frac{-b}{2a}$$

and plug it into the height equation.

EXAMPLE FOR GROUND LAUNCH

For the ground launch equation example, the students measure the hang time to be 4 seconds. Therefore,

$$v_0 = -5 \cdot t_h = -5 \cdot 4 = 20\, m/s$$

The students build their height equation.

$$H(t) = -5t^2 + v_0 t = -5t^2 + 20t$$

The students solve for half the hang time. For teachers creating a table for this lab, a column should be given for this, even though the calculation is straightforward.

$$t_{max} = 2$$

The students solve for maximum height.

$$H_{max} = -5\left(0.5t_h\right)^2 + 20\left(0.5t_h\right) = -5(2)^2 + 20(2)$$

$$H_{max} = -20 + 40 = 20\, meters$$

The students repeat the process using all their data.

EXAMPLE FOR PLATFORM LAUNCH

For the platform launch equation, the students measure the hang time and calculate the initial velocity. For this example, the student launches the projectile from 1 meter height and the hang time is 3.07 seconds. The student calculates the initial velocity:

$$v_0 = -\frac{h_0}{t_h} - t_h \cdot \frac{g}{2} = -\frac{1}{3.07} + 3.07 \cdot 5$$

$$v_0 = -0.33 + 15.35 = 15.02$$

The precise answer is 15; the teacher (or more likely the stopwatch) can determine the level of precision.

The students build their equation.

$$H(t) = -5t^2 + v_0 t + h_0 = -5t^2 + 15t + 1$$

The students solve for t_{max}.

$$t_{max} = \frac{-v_0}{g} = \frac{-15}{-10} = 1.5$$

The students solve for H_{max}.

$$H_{max} = -5(1.5)^2 + 15(1.5) + 1 = 12.25$$

The students repeat the process for all their data points.

MAHALO!

I hope you and your students enjoy this easy and convenient lab.