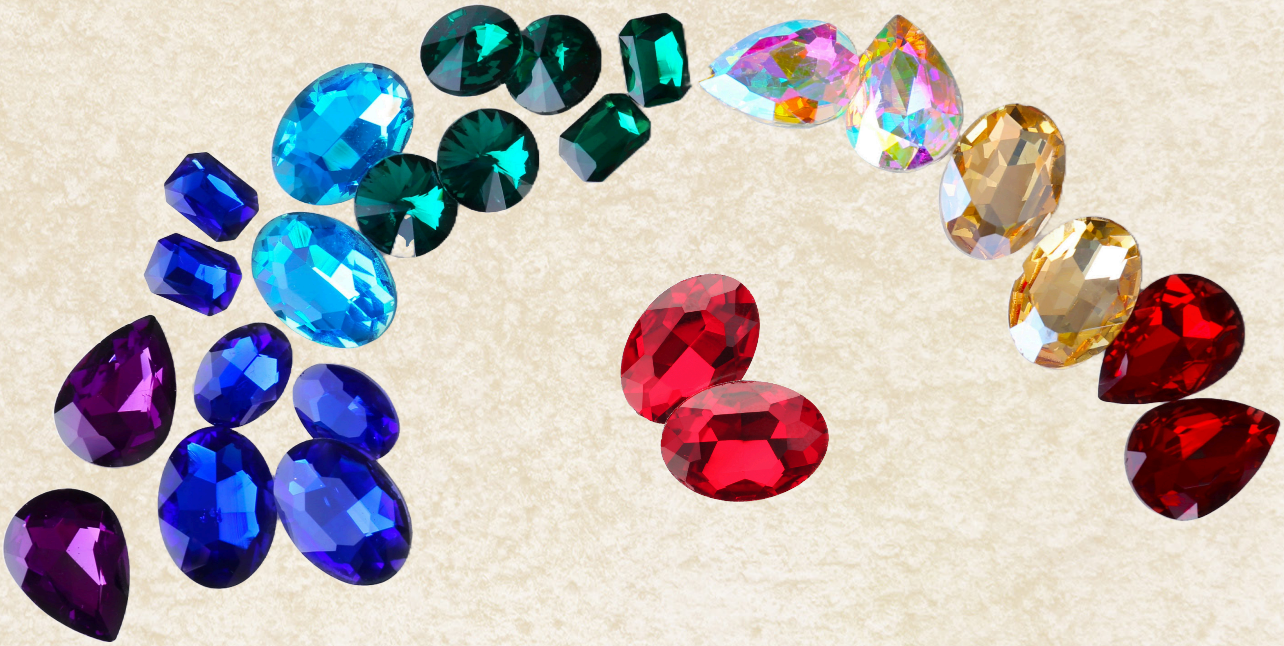
Mathematica Exploratio II



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MATHEMATICA EXPLORATIO II

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Special Thank You

to Christian Williams for evaluating this work

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INTRODUCTION

This paper focuses on methods for finding the roots of polynomials. As such it complements a number of my previous papers, namely Mathematica Exploratio I (2021), The 3-Space Method for Solving Parabolic Roots (2021), Ten Novel Methods for Solving Quadratic Roots (2023), and Delta Method for Solving Quadratic Equations and Special Cubic Equations (2024).

One significant feature to point out concerning the methods in this paper is that most of them are only functional for use with integer solutions; the exceptions are the Emerald Method and the Pink Diamond Method.

The methods and discoveries in this paper represent my own work and are original to the best of my knowledge.

Mathematica Exploratio I includes the 2C Quadratic Formula, which I prove geometrically in two different ways: first via the use of a secant and second via the use of similar triangles. I have recently learned of Muller's Method and the form of the quadratic formula that it employs to solve for roots. The expression is very similar to the 2C Quadratic Formula. Muller's Method dates to 1956, so he certainly arrived at the formula before I did, since I wasn't even born.

THE EMERALD METHOD

The Emerald Method solves parabolic roots by using the elimination method commonly employed in systems of equations. The mathematics is Algebra I or Algebra II level. *This method does reduce to the quadratic formula;* for this reason, I begin with examples so that the reader may see that its form in application is distinctly different. Ultimately, the method must be tested with students to see if it has educational value.

Because the method reduces to the quadratic formula, the reader may justifiably ask, “What’s the point?” I believe the derivation is mathematically interesting. Rather than “completing the square,” this method represents the difference of squares. The geometric visualization is different from that of completing the square and from the various geometric methods of solving quadratic roots that I previously developed: the delta method, vertex method, big sister method, black box method, twin method, companion method, or the three-dimensional methods.

Additionally, the prime solution anomaly is of interest. I was very surprised to discover this while running numbers through the equations.

EXAMPLES & SOLUTIONS

The first two examples are simple integer solutions; the third example includes decimal solutions; the fourth example is non-monic quadratic.

EXAMPLE 1. This example employs simple integer solutions.

Given: a monic parabola expressed in the form

$$y = x^2 + 13x + 36$$

Using two equations, set their values to *b* and the square root of the discriminant *d*.

$$s + t = 13$$

$$s - t = \sqrt{169 - 4 \cdot 36} = \sqrt{25} = 5$$

Use elimination. The first solution will be given by

$$\begin{array}{c} s + t = 13 \\ s - t = 5 \\ \hline 2s = 18 \end{array}$$

$$s = 9 \rightarrow s_* = -9$$

$$t = 13 - 9 = 4 \rightarrow t_* = -4$$

or equivalently for t by equation subtraction

$$\begin{array}{c} s + t = 13 \\ s - t = 5 \\ \hline 2t = 8 \end{array}$$

$$t = 4 \rightarrow t_* = -4$$

EXAMPLE 2. This example likewise employs simple integer solutions.

Given: a monic parabola expressed in the form

$$y = x^2 + 11x + 18$$

Using two equations, set their values to *b* and the square root of the discriminant *d*.

$$s + t = 11$$

$$s - t = \sqrt{121 - 4 \cdot 18} = \sqrt{49} = 7$$

Use elimination. The first solution will be given by

$$\begin{array}{c} s + t = 11 \\ s - t = 7 \\ \hline 2s = 18 \end{array}$$

$$s = 9 \rightarrow s_* = -9$$

$$t = 11 - 9 = 2$$

or equivalently for t by equation subtraction

$$\begin{array}{c} s + t = 11 \\ s - t = 7 \\ \hline 2t = 4 \end{array}$$

$$t = 2 \rightarrow t_* = -2$$

EXAMPLE 3. This example includes decimal solutions.

Given: a monic parabola expressed in the form

$$y = x^2 + 12x + 18$$

Now using two equations, set their values to b and the square root of d.

$$s + t = 12$$

$$s - t = \sqrt{144 - 4 \cdot 18} = \sqrt{72} = 8.485$$

Use elimination. The first solution will be given by

$$s + t = 12$$
$$s - t = 8.485$$

$$2s = 20.485$$

$$s = 10.243 \rightarrow s_* = -10.243$$

$$t = 12 - 10.243 = 1.757 \rightarrow t_* = -1.757$$

EXAMPLE 4. This example is non-monic. Note the modification. It is not very convenient as the reader can see. I will demonstrate an alternate method below.

Given: a non-monic parabola expressed in the form

$$y = -5x^2 + 13x + 36$$

Now using two equations, set their values to b and the square root of d. Both equations are multiplied by a.

$$a(s + t) = 13$$

$$a(s - t) = \sqrt{169 - 4 \cdot (-5) \cdot 36} = \sqrt{889} = 29.816$$

Use elimination.

solve for s with addition of equations

$$a(s + t) = 13$$
$$a(s - t) = 29.816$$

$$10s = 42.816$$

$$s = 4.2816 \rightarrow s_* = -4.2816$$

solve for t with subtraction of equations

$$a(s + t) = 13$$
$$a(s - t) = 29.816$$

$$10t = 13 - 29.816 = -16.816$$

$$t = -1.6816 \rightarrow t_* = 1.6816$$

NON-MONIC APPLICATION

Since the result of the Emerald Method is always the same (which is to say that adding or subtracting equations always results in $2s$ or $2t$), it is easier to solve non-monic quadratics in the following way. I use the example quadratic from above.

EXAMPLE 5. This example is non-monic.

Given: a non-monic parabola expressed in the form

$$y = -5x^2 + 13x + 36$$

$$b = 13$$

$$d = \sqrt{169 - 4 \cdot (-5) \cdot 36} = \sqrt{889} = 29.816$$

$$2as = 13 + 29.816$$

$$10s = 42.816$$

$$s = 4.2816 \rightarrow s_* = -4.2816$$

$$2at = 13 - 29.816$$

$$10t = -16.816$$

$$t = -1.6816 \rightarrow t_* = 1.6816$$

With this example, the reader can clearly see that this method reduces to the quadratic formula. However, there is one benefit, namely that in the physics representation, the velocities $2as$ and $2at$ are clearly apparent, since $2a$ is the actual acceleration. For example, we often use $a = -4.9$ for gravity as required by the physics, but $2a = -9.8$ displays its actual value. Because s and t are in the units of time, the result is that $2as$ and $2at$, like b and $\sqrt{d}$, are in the units of velocity.

EMERALD METHOD: IMAGE

$$y = x^2 + 8x + 15$$

$$x + y = 8$$

$$x - y = 2$$

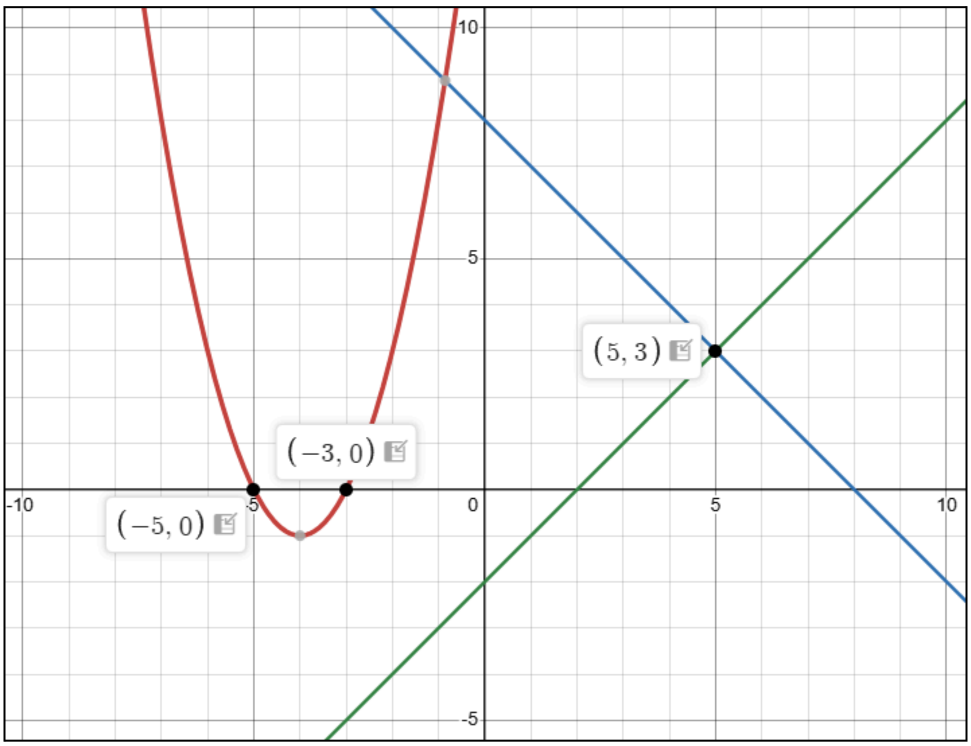


Image by author. Powered by Desmos.

DERIVATION

Part A. The Lengths.

Consider the parabola given by

$$y = ax^2 + bx + c$$

Let *b* be given as the sum of two solutions. For convenience, we will consider the terms to be integers and the first term to be greater than the second term.

$$s > t$$

$$b = s + t$$

Now consider a smaller square, which is the difference of the same terms.

$$d = s - t$$

Part B. The Squares.

We will now square both terms.

$$b^2 = (s + t)^2$$

$$d^2 = (s - t)^2$$

It is evident, given our allowed assumptions, that the former square has greater area than the latter square.

Part C. The Difference of Squares.

We solve for the difference of the squares.

$$b^2 - d^2 = \left(s^2 + 2st + t^2\right) - \left(s^2 - 2st + t^2\right) = 4st$$

The reader will take note: The result is identical to the constant *c* of the parabola.

$$c = st$$

Part D. Defining the Determinant Square Root.

Solving for d, we have

$$b^2 - d^2 = 4c$$

$$d = \sqrt{b^2 - 4c}$$

Therefore, d is the square root of the determinant of the quadratic formula.

Part E. Solving for d.

$$d = \sqrt{\left(s^2 + 2st + t^2\right) - 4st}$$

$$d = \sqrt{s^2 - 2st + t^2}$$

$$d = \sqrt{(s - t)^2}$$

$$d = s - t$$

This demonstrates consistency.

Part F. Applying to the Quadratic Formula.

To simplify this explanation, we use the monic form. Observe the quadratic formula in this form:

$$2 \times solutions = b \pm \sqrt{b^2 - 4c}$$

Applying our previous definitions, we can write

$$2 \times solutions = (s + t) \pm (s - t) = b \pm \sqrt{b^2 - 4c}$$

Thus, s and t are the solutions to the quadratic formula.

$$s + t \pm s - t = b \pm d$$

This is the form of which I demonstrated with examples.

Part G. Geometric Interpretation.

We have observed the equation

$$b^2 - d^2 = 4c$$

We may rewrite this equation in any of these forms:

$$b^2 = d^2 + 4c$$

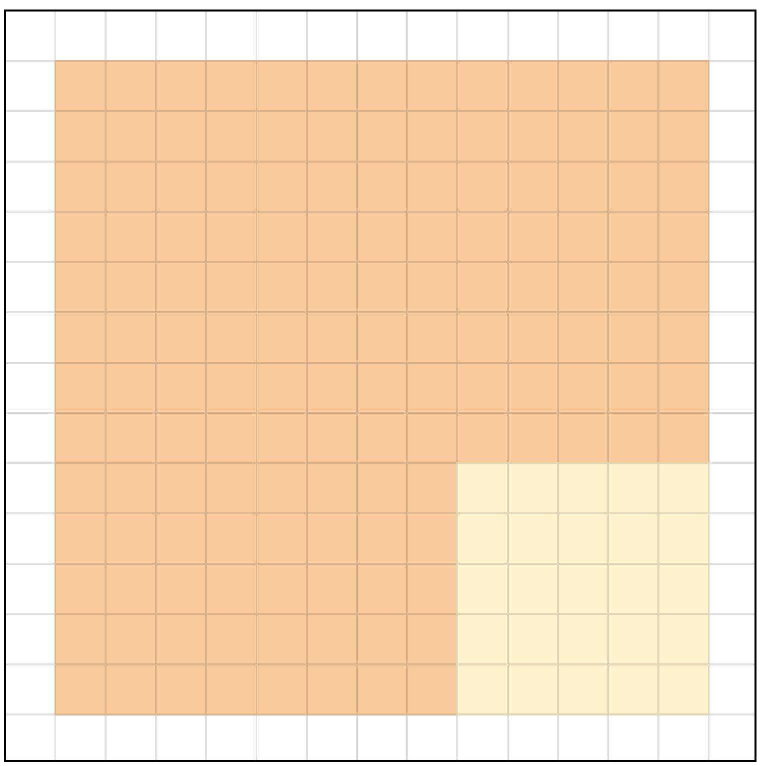
$$(s + t)^2 = (s - t)^2 + 4st$$

$$big\ square\ =\ small\ square\ +\ side\ area$$

This is interesting because it allows us to see c as more than just the y-intercept.

Part H. Sample Image.

$$y = x^2 + 13x + 36,\ s = 9,\ t = 4,\ b^2 = 169,\ d^2 = 25,\ 4st = 144$$



The complete square is b^2*, the yellow square is* d^2*, and the orange area is* $4st$*.*

Image by author.

PRIME SOLUTION ANOMALY

Refer to the equation and its rearrangement:

$$(s + t)^2 = (s - t)^2 + 4st$$

$$(s + t)^2 - 4st = (s - t)^2$$

Now for any given parabola, if the value

$$c = st$$

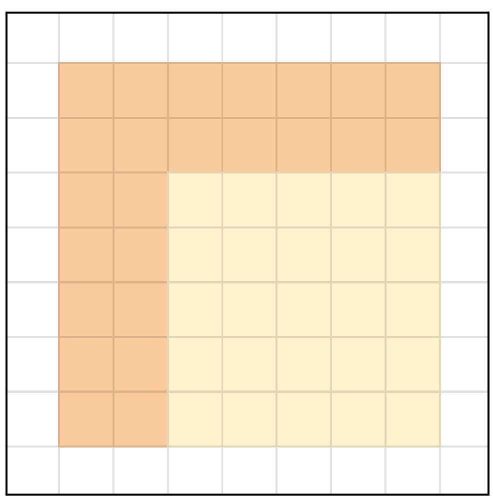
is a prime, the only possible integer value for b must be

$$b = c + 1$$

because the only solutions for a prime are the identity and itself. This is why I refer to it as the prime solution anomaly. Therefore, we give t the value of the identity and our equation becomes

$$(s + 1)^2 - 4s = (s - 1)^2$$

This is an intriguing result. Here are two examples:



$$7^2 - (4 \cdot 6) = 5^2$$

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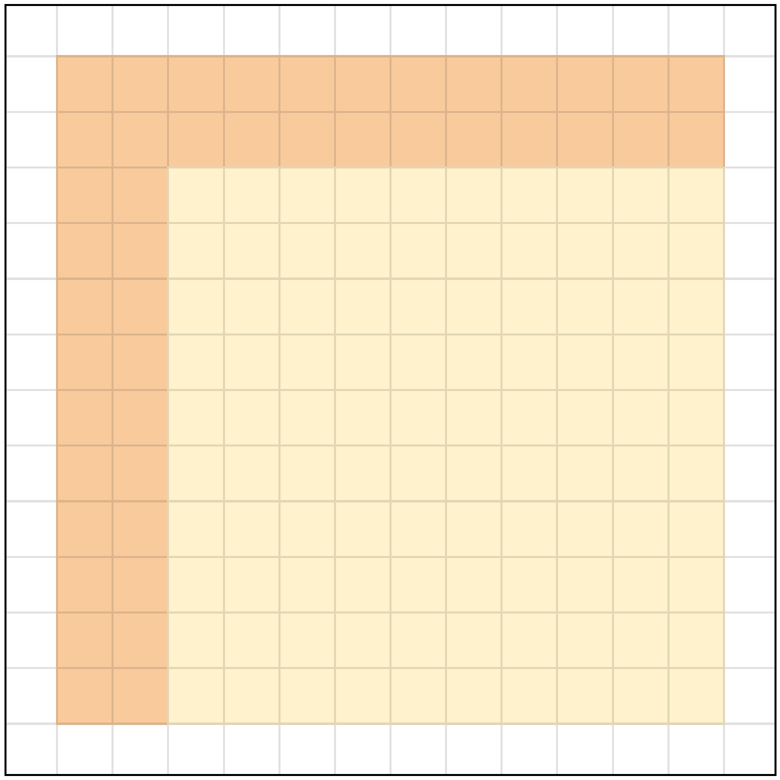

$$12^2 - (4 \cdot 11) = 10^2$$

Image by author.

The images are identical in form, but not in magnitude. It is most interesting to see that, regardless of the magnitude, four times one less than the base of the first term value equals the square of two less than the base of the first term value.

$$10\,000^2 - (4 \times 9999) = 9998^2$$

Intriguing!

THE SAPPHIRE METHOD

FOR SOLVING CUBIC EQUATIONS

The Sapphire Method solves cubic equations by reference to the prime solution (using the term “prime solution” as mentioned in the previous section). Here are our terms:

$$y = x^3 + ax^2 + bx + c$$

$$a = q + r + s$$

$$b = qr + rs + sq$$

$$c = qrs$$

Geometrically, a represents the sum of the lengths of the three distinct edges of the cube, b represents the sum of the area of the three distinct faces of the cube, and c represents the volume of the cube.

The Sapphire Method looks at the volume of the prime solution and compares it to the volume of the target equation. This allows the problem solver to estimate values are needed for r and s to multiply the prime solution volume such that it exceeds the target equation volume, such that, when that volume is reduced by the proportional reduction of q, will give the target volume. If this sounds a little confusing, I will teach the method by walking through it, but first we will look at the easiest cubic equations to solve, the identity equations.

THE IDENTITY ANOMALIES

Consider the following equations and their solutions:

$$y_P = x^3 + 12x^2 + 21x + 10 \quad q, r, s = 10,\ 1,\ 1$$

$$y_2 = x^3 + 12x^2 + 29x + 18 \quad q, r, s = 9,\ 2,\ 1$$

$$y_3 = x^3 + 12x^2 + 35x + 24 \quad q, r, s = 8,\ 3,\ 1$$

$$y_4 = x^3 + 12x^2 + 39x + 28 \quad q, r, s = 7,\ 4,\ 1$$

$$y_5 = x^3 + 12x^2 + 41x + 30 \quad q, r, s = 6,\ 5,\ 1$$

I refer to these as “identity equations” because they each contain 1 as a solution. The reader will note the following three anomalies of these equations:

$$b_n - c_n = a_n - 1$$

$$b_n - b_P = c_n - c_P$$

$$b_{n+k} - b_n = c_{n+k} - c_n$$

The first anomaly is simple to prove:

$$where\ s = 1$$

$$a = q + r + 1$$

$$b = qr + r + q$$

$$c = qr$$

$$\therefore$$

$$b - c = qr + r + q - qr = q + r = a - 1$$

The second and third anomalies (which are really equivalent) can be proven by incrementation. We will prove by example using y_2 and y_3.

$$b_2 = (q_P - 1)(r_P + 1) + (r_P + 1)(s_P) + (s_P)(q_P - 1)$$

$$b_2 = q_P r_P - r_P + q_P - 1 + r_P s_P + s_P + s_P q_P - s_P$$

$$b_2 = q_P r_P + r_P s_P + s_P q_P - r_P + q_P + s_P - s_P - 1$$

$$b_2 = q_P r_P + r_P s_P + s_P q_P - r_P + q_P - 1$$

$$where\, r_P = s_P = 1$$

$$b_2 = 3q_P - 1$$

$$b_3 = (q_P - 2)(r_P + 2) + (r_P + 2)(s_P) + (s_P)(q_P - 2)$$

$$b_3 = q_P r_P - 2r_P + 2q_P - 4 + r_P s_P + 2s_P + q_P s_P - 2s_P$$

$$b_3 = q_P r_P + r_P s_P + q_P s_P - 2r_P + 2q_P + 2s_P - 2s_P - 4$$

$$b_3 = q_P r_P + r_P s_P + q_P s_P - 2r_P + 2q_P - 4$$

$$where\, r_P = s_P = 1$$

$$b_3 = q_P + 1 + q_P - 2 + 2q_P - 4$$

$$b_3 = 4q_P - 5$$

$$b_3 - b_2 = (4q_P - 5) - (3q_P - 1)$$

$$RESULT:\ b_3 - b_2 = q_P - 4$$

$$c_2 = \left(q_P - 1\right)\left(r_P + 1\right)s_P = \left(q_p - r_P + q_P r_P - 1\right) \cdot s_P$$

$$c_3 = \left(q_P - 2\right)\left(r_P + 2\right)s_P = \left(2q_p - 2r_P + q_P r_P - 4\right) \cdot s_P$$

$$where\, r_P = s_P = 1$$

$$c_2 = \left(q_p - 1 + q_P - 1\right) \cdot 1 = 2q_P - 2$$

$$c_3 = \left(2q_p - 2 + q_P - 4\right) \cdot 1 = 3q_P - 6$$

$$c_3 - c_2 = \left(3q_P - 6\right) - \left(2q_P - 2\right)$$

$$RESULT:\ c_3 - c_2 = q_P - 4$$

$$\triangle\, b_3 - b_2 = c_3 - c_2$$

Solving identity equations is simple. The problem solver must first identify the cubic as an identity equation by noting that $b - c = a - 1$. Therefore, $s = 1$. The problem solver need only solve for q and r. This can be done through the quadratic:

$$y = x^2 + (a - 1)x + b$$

EXAMPLE

$$y = x^3 + 18x^2 + 89x + 72$$

The reader notices that

$$89 - 72 = 17$$

$$18 - 1 = 17$$

We have a unity cubic! The reader can now solve

$$y = x^2 + 17x + 72$$

$$q = 9,\ r = 8$$

Complete.

SAPPHIRE SOLVING

The benefit of the Sapphire Method is that it reduces or even eliminates guessing from solving cubic equations. I will teach this method by walking through a sample problem.

$$y = x^3 + 12x^2 + 41x + 42$$

Step 1: ***Find the prime equation.***

$$a = 10 + 1 + 1 \quad q,\ r,\ s = 10,\ 1,\ 1$$

$$y_P = x^3 + 12x^2 + \gamma x + 10$$

It is not necessary to solve for b_P; this is why I use γ in its place.

Step 2: ***Line up the two equations and include the solutions for y_P.***

$$y_P = x^3 + 12x^2 + \gamma x + 10 \quad 10\ 1\ 1$$

$$y = x^3 + 12x^2 + 41x + 42 \quad -- - -$$

Step 3: ***Estimate the multiple volume increase.***

Our plan is to find values for r and s by increasing r_P and s_P. Any increase in r_P and s_P means an increase in volume. Since $r_P = s_P = 1$, we need only multiply r and s by q_P to find the multiplied volume. The new volume will be $q_P \cdot r \cdot s$. But wait! If we increase r_P and s_P , don't we have to decrease q_P by the same amount? Yes, but we'll do that afterwards.

By observation, we know that quadrupling the volume will not be enough. That would only give us a volume of 40. We need to shoot for a volume of at least 50. This only allows one possibility for r and s: $r = 5$ and $s = 1$. But we know the cubic is not an identity equation, so that won't work. Therefore, we try $r = 3$ and $s = 2$. This will give us a volume of 60. That's too much! But the solution may work because this volume will be reduced by the reduction in q_P.

Step 4: ***Find q***.

Here δr_P and δs_P represent what you increased r_P and s_P by. Essentially, $\delta r_P = r - 1$ and $\delta s_P = s - 1$.

$$r = 3 \quad \delta r_P = 2$$

$$s = 2 \quad \delta s_P = 1$$

$$q = q_P - (\delta r + \delta s) = 10 - 3 = 7$$

Step 5: ***Apply the estimates.***

$$y_P = x^3 + 12x^2 + \gamma x + 10 \quad 10\ 1\ 1$$

$$y = x^3 + 12x^2 + 41x + 42 \quad 7\ 3\ 2$$

The reader can see that the answer is correct:

$$a = 7 + 3 + 2 = 12$$

$$c = 7 \cdot 3 \cdot 2 = 42$$

EXAMPLE:

$$y = x^3 + 12x^2 + 45x + 50$$

Step 1 & 2: ***Find the prime equation and line the two up.***

$$y_P = x^3 + 12x^2 + \gamma x + 10 \quad 10\ 1\ 1$$

$$y = x^3 + 12x^2 + 45x + 50 \quad -- - -$$

Step 3 & 4: ***Estimate the multiple volume increase.***

This volume approaches the cube. Yet the cube volume equals $\left(\frac{12}{3}\right)^3 = 64$. Therefore, the solutions are not $(4,\ 4,\ 4)$. But we can try the closest approximate $(5,\ 5,\ 2)$.

Step 5: ***Apply the estimates*****.**

$$y_P = x^3 + 12x^2 + \gamma x + 10 \quad 10\ 1\ 1$$

$$y = x^3 + 12x^2 + 45x + 50 \quad 5\ 5\ 2$$

The reader can see that

$$a = 5 + 5 + 2 = 12$$

$$c = 5 \cdot 5 \cdot 2 = 50$$

Complete!

THE PINK DIAMOND METHOD

This method works for solving quadratic equations when $b > c, a = 1$, and $b, c > 0$. Consider the quadratic equation in the general monic form:

$$y = x^2 + bx + c$$

Let the solutions be given as q and r. The solutions set will be

$$(q,\ r = b - q)$$

Thus, we need only solve for q.

We begin this method with an approximation that takes this form:

$$q = \frac{c}{b} + \frac{c^2}{b^3}$$

I will demonstrate the accuracy of this approximation with a number of examples in the following table. The third column represents the monic form of the quadratic formula.

EQUATION	$q = \frac{c}{b} + \frac{c^2}{b^3}$	$q = \frac{1}{2}\left(-b + \sqrt{b^2 - 4c}\right)$
$y = x^2 + 150x + 11$	-0.7336918	-0.7336922
$y = x^2 + 15x + 1$	-0.0669630	-0.0669656
$y = x^2 + 40x + 5$	-0.1253906	-0.1253931
$y = x^2 + 40x + 20$	-0.5062500	-0.5064110
$y = x^2 + 40x + 30$	-0.7640625	-0.7641594

The learner will notice that the greater accuracy corresponds to the greater difference between b and c.

To discover the source of this phenomenon, values of b and c are selected that best demonstrate the numerical workings.

$$y = x^2 + 10x + 1$$

$$q = -5 + 2\sqrt{6} = -0.10102051443364380364380 36$$

$$y = x^2 + 100x + 1$$

$$q = -50 + 7\sqrt{51} = -0.010001000200050014004201 32$$

The number sequence is:

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132$$

This is the Catalan Sequence designated A000108. The equation is

$$C(n) = \frac{(2n)!}{n!(n+1)!}$$

This equation only gives the numeric value of the numerator of the sequence. For the solution q to the quadratic, the complete equation is

$$\sum_{n=0}^{\infty} = \frac{(2n)!}{n!(n+1)!} \cdot \frac{1}{b^{2n+1}}$$

I believe this is correct. It's worth noting that the same numeric series, with opposite signs, occurs when $b < 0$.

$$y = x^2 - 100x + 1$$

$$q = 50 - 7\sqrt{51} = 0.010001000200050014004201 32$$

Let's extend our observations to quadratic equations where $c \neq 1$.

$$y = x^2 + 10x + 2$$

$$q = -5 + \sqrt{23} = -0.20416847668728045840256 19$$

$$y = x^2 + 100x + 2$$

$$q = -50 + \sqrt{2498} = -0.0200040016008004482689690699$$

The number sequence is:

$$2,\ 4,\ 16,\ 80,\ 448,\ 2688$$

This is the Catalan Sequence designated A025225. The equation is

$$a(n) = 2^n C(n - 1)$$

The reader can observe the multiplicative relationship in this table:

$C(n)$	1	1	2	5	14	42
2^n	2	4	8	16	32	64
$a(n)$	2	4	16	80	448	2688

Therefore, our general expression should be

$$\sum_{n=0}^{\infty} = \frac{(2n)!}{n!(n+1)!} \cdot \frac{c^n}{b^{2n+1}}$$

Thus we have the equation and the solutions

$$y = x^2 - 100x + 2$$

$$q = \frac{2}{100} + \frac{4}{100^3} + \frac{16}{100^5} + \frac{80}{100^7} \cdots$$

$$r = b - q$$

Recall that where b is negative, q and r are positive; where b is positive, q and r are negative.

Please see TOOLS at the end of this work to reference the tools which I used for the quadratic formula solutions, precision values, and sequence identifications for the Pink Diamond Method.

Let's return to demonstrating in a typical classroom setting the use of the approximation with an example problem:

$$y = x^2 + 12x + 3$$

$$q = -\frac{3}{12} - \frac{3^2}{12^3} = -0.2552$$

$$r = -12 + 0.2552 = -11.7448$$

These answers are accurate to the 10,000th place.

THE RUBY METHOD

I will teach by example beginning with a cubic.

$$y = x^3 + 22x^2 + 141x + 252$$

The prime factorization of the volume is

$$252 = 2^2 \cdot 3^2 \cdot 7$$

Begin by creating a table. The table must begin with the largest prime.

$a = 22$	q	r	s	$c = 252$	remaining
9	7	1	1	7	2, 2, 3, 3

We are looking to add to 22 and multiply to 252. Therefore, we know that we cannot multiply the first slot by 3 because 21 would bring the sum to 23. Therefore, we can place 3 in a slot.

$a = 22$	q	r	s	$c = 252$	remaining
11	7	3	1	21	2, 2, 3

We cannot multiply q by 2 because this would mean $a = 18$ and applying the remaining prime factors would exceed 22. So $q = 7$. If we multiply r by 3 (and let $s = 4$), then $a = 20$, which does not meet our required 22. So we must place 3 in the s column.

$a = 22$	q	r	s	$c = 252$	remaining
13	7	3	3	63	2, 2

If we split the 2s into the r and s columns, we only increase a by 6 so that $a = 19$. This is too low. So both 2s belong in a single column.

$a = 22$	q	r	s	$c = 252$	remaining
22	7	12	3	252	$\emptyset$

Complete!

THE AMETHYST METHOD

I will teach by example using a quintic.

$$y = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

$$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$$

We are only interested in $a = 29$ and $e = 4536$.

Create a table that contains all the prime factors. There are 5 empty cells on the bottom of the table, one for each factor. In the Amethyst Method, the product remains a constant. We merely need to combine the factors so that they add to the required sum.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$								$e = 4536$
25	2	2	2	3	3	3	3	7	4536
									4536

The chart tells us that we are missing 4 in the sum. One interesting rule: Combining 2 and 2 does not add to the sum, since $2 + 2 = 2 \times 2$. Of course, we need to be aware that 1 may be a factor.

Let's begin! Combining 3 and 3 gives 9, which adds 3 to the sum. We only need an additional 1. Combining 2 and 3 gives 6, which adds 1 to the sum. We can combine the remaining 2s together without cost and leave the 3 alone.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$								$e = 4536$
25	2	2	2	3	3	3	3	7	4536
29			4	6	3	9	7		4536

Complete!

The Amethyst Method may be a convenient method for high school algebra students solving quadratics that have large numbers. Prime factors can be found using a factor tree.

EXAMPLE:

$$y = x^2 + 22 + 96$$

$a = 22$	$y = x^2 + 22 + 96$								$b = 96$
13		2	2	2	2	2	3		96

Our sum is 13, but we need 22. We need an additional 9 to the sum. Combining 2 and 2 to get 4 adds 0, and that leaves $2 \times 2 \times 2 \times 3 = 48$ – way too high. We can try 8 and 12; they add to 20. Not quite. What about 16 and 6? They add to 22.

$a = 22$	$y = x^2 + 22 + 96$								$b = 96$
13		2	2	2	2	2	3		96
22				16	6				96

Complete!

An interesting method that perhaps has its uses.

THE TURQUOISE METHOD

I will teach by example using a quintic.

$$y = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

$$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$$

We are only interested in $a = 29$ and $e = 4536$.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$

Let the solutions be given by *q, r, s, t, u*. The largest possible value for *q* is 25. (This would be for the solution set $(25, 1, 1, 1, 1)$.) But 4536 is not divisible by 25, so we shall use 24 instead.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$
213		24	189			4536

It is unlikely that 24 is a solution, but let's leave it alone for now. Clearly, 189 is too large and needs to be broken down. We could break 189 to 3 and 63, but this is not ideal. We don't know if 3 is a solution and 63 is still too big for our target sum 29. So let's break 189 into 9 and 21.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$
213		24	189			4536
54		24	9	21		4536

We have only two open slots for solutions. We could break the 9 into 3 and 3, but this is unlikely to work, since keeping the 24 or 21 will likely put our sum over 29. Therefore, 9 is a solution.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$
213		24	189			4536
54		24	9	21		4536

To reach our target 29, both the 21 and 24 need to be broken down. Let's first break down the 21 because it breaks into two primes.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$
213		24	189			4536
54		24	9	21		4536
43		24	9	3	7	4536

Now we are 14 above our target sum of 29 (because $43 - 29 = 14$). This means that we have to reduce the 24 by 14. Therefore, the factors of 24 must sum to 10.

$a = 29$	$y = x^5 + 29x^4 + 325x^3 + 1755x^2 + 4554x + 4536$					$e = 4536$
213		24	189			4536
54		24	9	21		4536
43		24	9	3	7	4536
29	6	4	9	3	7	4536

Complete!

TOOLS

USED FOR THE PINK DIAMOND METHOD

Furey, Edward. "Quadratic Formula Calculator" at https://www.calculatorsoup.com/calculators/algebra/quadratic-formula-calculator.php from CalculatorSoup, https://www.calculatorsoup.com - Online Calculators. Update version: 2025 Feb. 24.

OEIS Foundation. The On-Line Encyclopedia of Integer Sequences. Update version: 2025 June 23. https://oeis.org/

Tommila, Mikko. Online Arbitrary Precision Calculator. Update version: 2022 Jan. 4. https://www.apfloat.org/calculator/