Explorations *on the* Euler Constants

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first paper

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April 2025

CONTENTS

THE NATURAL LOGARITHM AS A FUNCTION OF A SUM: The Λ Approximation

From the di-gamma function, we have:

$$\psi^{(0)} = -\gamma_e + \sum_{n=1}^{z-1} \frac{1}{n}$$

$$\psi^{(0)} \approx ln(z) - \frac{1}{2z}$$

We have the Euler gamma constant defined as

$$\gamma_e = 0.57721566$$

We combine the two equations:

$$ln(z) \approx \Lambda = \sum_{n=1}^{z-1} \frac{1}{n} + \frac{1}{2z} - \gamma_e$$

We denote the natural log approximation as Λ. The following Python code is programmed to compare this approximation with the natural log.

```
## Analyzing ln as a sum ##
import math

counter = 0 ## Initiate
z = 2
sigma_sum = 0
gamma = 0.5772156649
big_sum = 0

while counter < 25:
    sigma_sum = sigma_sum + (1 / (z-1))
    big_sum = sigma_sum + (1 / (2 * z)) - gamma
    ln = math.log(z)
    print (z, "\t", f"{ln:.10f}", "\t",
          f"{big_sum:.10f}", "\t",
          f"{sigma_sum:.10f}", "\t", f"{(1 /
          (2*z)):.10f}", "\t", -0.57721566)
    z = z + 1
    counter = counter + 1
```

To analyze the accuracy of the approximation, we run the code. Note that the first column is z, the second is the natural log, the third is the approximation and the fourth, fifth, and sixth are the terms of the approximation in the order they appear in the equation above.

z	ln	Λ	$\sum_{n=1}^{z-1} \frac{1}{n}$	$\frac{1}{2z}$	γ_e
2	0.69314718	0.672784340	1.000000000	0.250000000	-0.57721566
3	1.09861229	1.089451007	1.500000000	0.166666667	-0.57721566
4	1.38629436	1.381117673	1.833333333	0.125000000	-0.57721566
5	1.60943791	1.606117673	2.083333333	0.100000000	-0.57721566
6	1.79175947	1.789451007	2.283333333	0.083333333	-0.57721566
7	1.94591015	1.944212911	2.450000000	0.071428571	-0.57721566
8	2.07944154	2.078141483	2.592857143	0.062500000	-0.57721566
9	2.19722458	2.196197038	2.717857143	0.055555556	-0.57721566
10	2.30258509	2.301752594	2.828968254	0.050000000	-0.57721566
11	2.39789527	2.397207139	2.928968254	0.045454546	-0.57721566
12	2.48490665	2.484328352	3.019877345	0.041666667	-0.57721566
13	2.56494936	2.564456557	3.103210678	0.038461539	-0.57721566
14	2.63905733	2.638632381	3.180133755	0.035714286	-0.57721566
15	2.70805020	2.707680000	3.251562327	0.033333333	-0.57721566
16	2.77258872	2.772263333	3.318228993	0.031250000	-0.57721566
17	2.83321334	2.832925098	3.380728993	0.029411765	-0.57721566
18	2.89037176	2.890114640	3.439552523	0.027777778	-0.57721566
19	2.94443898	2.944208208	3.495108078	0.026315790	-0.57721566
20	2.99573227	2.995523997	3.547739657	0.025000000	-0.57721566
21	3.04452244	3.044333521	3.597739657	0.023809524	-0.57721566
22	3.09104245	3.090870318	3.645358705	0.022727273	-0.57721566
23	3.13549422	3.135336721	3.690813250	0.021739130	-0.57721566
24	3.17805383	3.177909184	3.734291511	0.020833333	-0.57721566
25	3.21887582	3.218742518	3.775958178	0.020000000	-0.57721566
26	3.25809654	3.257973287	3.815958178	0.019230769	-0.57721566

NUMERIC ANALYSIS AND INCREASED PRECISION: THE Θ APPROXIMATION

We now run the program to display the difference between the natural log and the approximation. The δ is the difference by subtraction; its inverse is also given.

z	ln	Λ	δ	δ^{-1}
2	0.6931471806	0.6727843400	0.0203628455	49.1090502
3	1.098612289	1.0894510067	0.0091612869	109.1549703
4	1.386294361	1.3811176733	0.0051766927	193.17353
5	1.609437912	1.6061176733	0.0033202440	301.1826841
6	1.791759469	1.7894510067	0.0023084675	433.1878256
7	1.945910149	1.9442129114	0.0016972425	589.1909873
8	2.079441542	2.0781414829	0.0013000637	769.1930653
9	2.197224577	2.1961970384	0.0010275438	973.1945024
10	2.302585093	2.3017525940	0.0008325039	1201.195537
11	2.397895273	2.3972071394	0.0006881383	1453.196306
12	2.48490665	2.4843283515	0.0005783031	1729.196893
13	2.564949358	2.5644565567	0.0004928057	2029.197352
14	2.63905733	2.6386323808	0.0004249537	2353.197718
15	2.708050201	2.7076799999	0.0003702061	2701.198015
16	2.772588722	2.7722633332	0.0003253939	3073.198259
17	2.833213344	2.8329250979	0.0002882510	3469.198462
18	2.890371758	2.8901146404	0.0002571224	3889.198635
19	2.944438979	2.9442082077	0.0002307764	4333.198782
20	2.995732274	2.9955239971	0.0002082813	4801.19891
21	3.044522438	3.0443335210	0.0001889217	5293.199022
22	3.091042453	3.0908703175	0.0001721408	5809.199121
23	3.135494216	3.1353367207	0.0001575002	6349.19921
24	3.17805383	3.1779091844	0.0001446508	6913.199291
25	3.218875825	3.2187425178	0.0001333120	7501.199365
26	3.258096538	3.2579732870	0.0001232559	8113.199434

I have highlighted the inverted deltas which allow us to see the pattern. We will zero-in on

$$\delta_5^{-1} = 301.183$$

Utilizing this value, we can see that the pattern within the inverse deltas can be described by the sigma value:

$$\sigma_1 = \left(\frac{z}{5}\right)^2 \cdot \delta_5^{-1}$$

This allows us to adjust the equation of approximation. Let's denote the modified approximation as Θ.

$$\Theta = \sum_{n=1}^{z-1} \frac{1}{n} + \frac{1}{2z} + \frac{1}{\sigma_1} - \gamma_e$$

The Python code is modified to include this term.

```
## Analyzing ln as a sum ##
import math

counter = 0 ## Initiate
z = 2
sigma_sum = 0
gamma = 0.5772156649
big_sum = 0

while counter < 25:
    sigma_sum = sigma_sum + (1 / (z-1))
    big_sum = sigma_sum + (1 / (2 * z)) - gamma
    super_big_sum = big_sum + (1 / ((z / 5)**2 *
          301.1831285792))
    ln = math.log(z)
    print (z, "\t", f"{ln:.10f}", "\t",
          f"{super_big_sum:.10f}", "\t",
          f"{big_sum:.10f}")
    z = z + 1
    counter = counter + 1
```

This table displays the results of the natural log, the modified approximation Θ, and the initial approximation Λ. The reader will observe that the modification significantly improves the precision of the approximation.

z	ln	Θ	Λ
2	0.6931471806	0.6935358344	0.67278434
3	1.098612289	1.098673893	1.089451007
4	1.386294361	1.386305547	1.381117673
5	1.609437912	1.609437912	1.606117673
6	1.791759469	1.791756728	1.789451007
7	1.945910149	1.945906911	1.944212911
8	2.079441542	2.079438451	2.078141483
9	2.197224577	2.197221804	2.196197038
10	2.302585093	2.302582654	2.301752594
11	2.397895273	2.397893139	2.397207139
12	2.48490665	2.484904782	2.484328352
13	2.564949358	2.564947716	2.564456557
14	2.63905733	2.639055881	2.638632381
15	2.708050201	2.708048915	2.70768
16	2.772588722	2.772587575	2.772263333
17	2.833213344	2.833212316	2.832925098
18	2.890371758	2.890370832	2.89011464
19	2.944438979	2.944438141	2.944208208
20	2.995732274	2.995731512	2.995523997
21	3.044522438	3.044521743	3.044333521
22	3.091042453	3.091041817	3.090870318
23	3.135494216	3.135493632	3.135336721
24	3.17805383	3.178053292	3.177909184
25	3.218875825	3.218875327	3.218742518
26	3.258096538	3.258096077	3.257973287

The Θ equation is not quite correct; the use of δ_5^{-1} is good, but our next analysis will provide a better solution.

MODULAR ANALYSIS: THE Φ APPROXIMATION

Let us return to the Λ approximation and examine the inverse deltas. The reader will notice that, when rounded down to an integer, each inverse delta q takes the form

$$q = k \cdot 12 + 1 \quad \text{or} \quad q \equiv 1 \ (mod\ 12)$$

Primes are denoted with shading (where decimals are ignored).

z	ln	Λ	δ^{-1}	$q\ equation$
2	0.6931471806	0.6727843400	49.109	$49 = 4 \cdot 12 + 1$
3	1.098612289	1.0894510067	109.155	$109 = 9 \cdot 12 + 1$
4	1.386294361	1.3811176733	193.174	$193 = 16 \cdot 12 + 1$
5	1.609437912	1.6061176733	301.183	$301 = 25 \cdot 12 + 1$
6	1.791759469	1.7894510067	433.188	$433 = 36 \cdot 12 + 1$
7	1.945910149	1.9442129114	589.191	$589 = 49 \cdot 12 + 1$
8	2.079441542	2.0781414829	769.193	$769 = 64 \cdot 12 + 1$
9	2.197224577	2.1961970384	973.195	$973 = 81 \cdot 12 + 1$
10	2.302585093	2.3017525940	1201.196	$1201 = 100 \cdot 12 + 1$
11	2.397895273	2.3972071394	1453.196	$1453 = 121 \cdot 12 + 1$
12	2.48490665	2.4843283515	1729.197	$1729 = 144 \cdot 12 + 1$
13	2.564949358	2.5644565567	2029.197	$2029 = 169 \cdot 12 + 1$
14	2.63905733	2.6386323808	2353.198	$2353 = 196 \cdot 12 + 1$
15	2.708050201	2.7076799999	2701.198	$2701 = 225 \cdot 12 + 1$
16	2.772588722	2.7722633332	3073.198	$3073 = 256 \cdot 12 + 1$
17	2.833213344	2.8329250979	3469.198	$3469 = 289 \cdot 12 + 1$
18	2.890371758	2.8901146404	3889.199	$3889 = 324 \cdot 12 + 1$
19	2.944438979	2.9442082077	4333.199	$4333 = 361 \cdot 12 + 1$
20	2.995732274	2.9955239971	4801.199	$4801 = 400 \cdot 12 + 1$
21	3.044522438	3.0443335210	5293.199	$5293 = 441 \cdot 12 + 1$
22	3.091042453	3.0908703175	5809.199	$5809 = 484 \cdot 12 + 1$
23	3.135494216	3.1353367207	6349.199	$6349 = 529 \cdot 12 + 1$
24	3.17805383	3.1779091844	6913.199	$6913 = 576 \cdot 12 + 1$
25	3.218875825	3.2187425178	7501.199	$7501 = 625 \cdot 12 + 1$
26	3.258096538	3.2579732870	8113.199	$8113 = 676 \cdot 12 + 1$

This analysis makes clear the imperfection in the previous analysis. The improved adjustment is given by the sigma:

$$\sigma_2 = z^2 \cdot 12 + 1$$

The new equation, denoted by Φ, is

$$\Phi = \sum_{n=1}^{z-1} \frac{1}{n} + \frac{1}{2z} + \frac{1}{\sigma_2} - \gamma_e$$

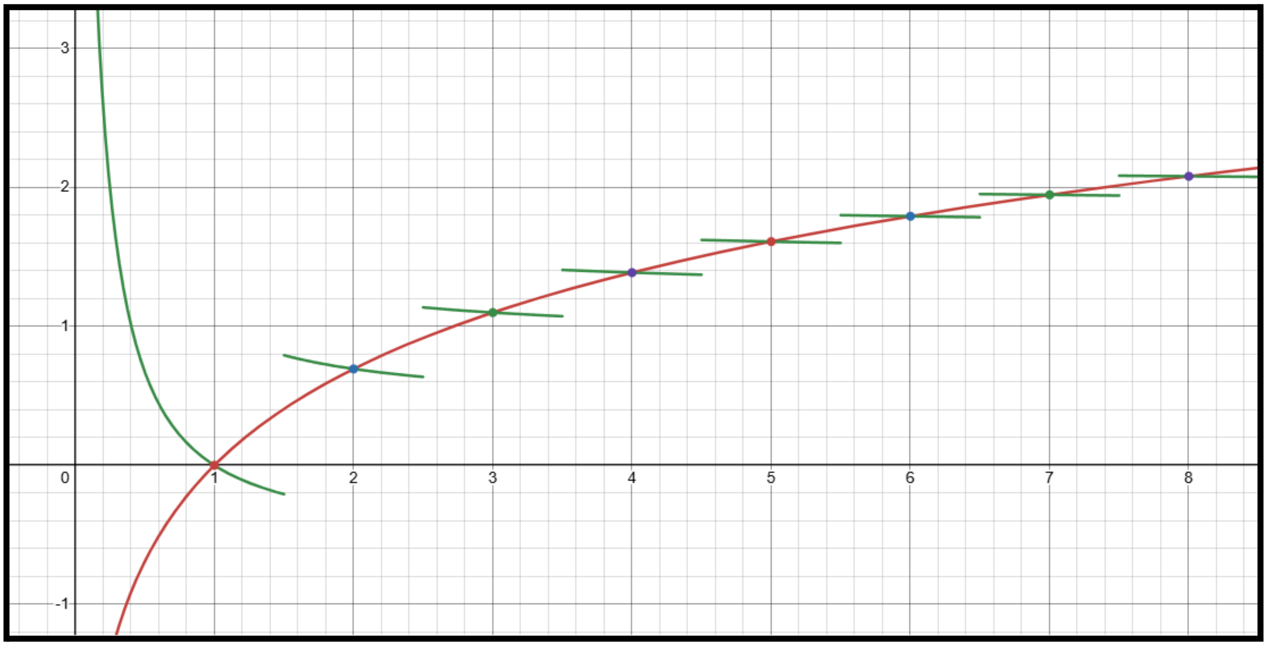
The Python code is augmented to include this improvement.

```
## Analyzing ln as a sum ##
import math

counter = 0 ## Initiate
z = 2
sigma_sum = 0
gamma = 0.5772156649
big_sum = 0

while counter < 25:
    sigma_sum = sigma_sum + (1 / (z-1))
    big_sum = sigma_sum + (1 / (2 * z)) - gamma
    mod_value = z**2 * 12 + 1
    super_mod_big_sum = big_sum + (1 / mod_value)
    super_big_sum = big_sum + (1 / ((z / 5)**2 *
          301.1831285792))
    ln = math.log(z)
    print (z, "\t", f"{ln:.10f}", "\t",
          f"{super_mod_big_sum:.10f}", "\t",
          f"{super_big_sum:.10f}", "\t",
          f"{big_sum:.10f}")
    z = z + 1
    counter = counter + 1
```

The table displays the improved precision of Φ with the greatest level of improvement in the initial z values.

z	ln	Φ	Θ	Λ
2	0.693147181	0.693192503	0.6935358344	0.67278434
3	1.098612289	1.098625319	1.098673893	1.089451007
4	1.386294361	1.386299021	1.386305547	1.381117673
5	1.609437912	1.609439933	1.609437912	1.606117673
6	1.791759469	1.791760476	1.791756728	1.789451007
7	1.945910149	1.945910704	1.945906911	1.944212911
8	2.079441542	2.079441873	2.079438451	2.078141483
9	2.197224577	2.197224788	2.197221804	2.196197038
10	2.302585093	2.302585233	2.302582654	2.301752594
11	2.397895273	2.397895371	2.397893139	2.397207139
12	2.484906650	2.484906721	2.484904782	2.484328352
13	2.564949358	2.564949410	2.564947716	2.564456557
14	2.639057330	2.639057370	2.639055881	2.638632381
15	2.708050201	2.708050233	2.708048915	2.70768
16	2.772588722	2.772588748	2.772587575	2.772263333
17	2.833213344	2.833213365	2.833212316	2.832925098
18	2.890371758	2.890371776	2.890370832	2.89011464
19	2.944438979	2.944438995	2.944438141	2.944208208
20	2.995732274	2.995732287	2.995731512	2.995523997
21	3.044522438	3.044522450	3.044521743	3.044333521
22	3.091042453	3.091042464	3.091041817	3.090870318
23	3.135494216	3.135494226	3.135493632	3.135336721
24	3.178053830	3.178053839	3.178053292	3.177909184
25	3.218875825	3.218875833	3.218875327	3.218742518
26	3.258096538	3.258096546	3.258096077	3.257973287

The approximation is modeled using the Desmos graphing calculator.



The natural log is displayed in red. The phi approximation is displayed in green. Image by author. Powered by Desmos.

REMARK ON THE MODULAR PHENOMENON

I have tested the output up to $z = 36$ and can confirm that all results conform to

$$q \equiv 1 \ (mod\ 12)$$

However long this pattern continues, it is well established. For example, where $z = 100$,

$$\delta_{100}^{-1} = 120001$$

which is clearly within $q = 12 \cdot k + 1$.

According to my investigations, the numbers of this subset come in two types: They are either the sum of squares or the product of primes, each prime occurring only once. I have tested this on all numbers $q \equiv 1 \ (mod\ 12)$ less than 400 and all numbers occurring in the subset $\sigma_2 = z^2 \cdot 12 + 1$ for all values of σ_2 up to 8113. In the first subset, the vast majority of numbers are a sum of squares; in the latter, the ratio is closer to equal.

FUNCTION FOR THE *n* APPROXIMATION OF THE EULER GAMMA

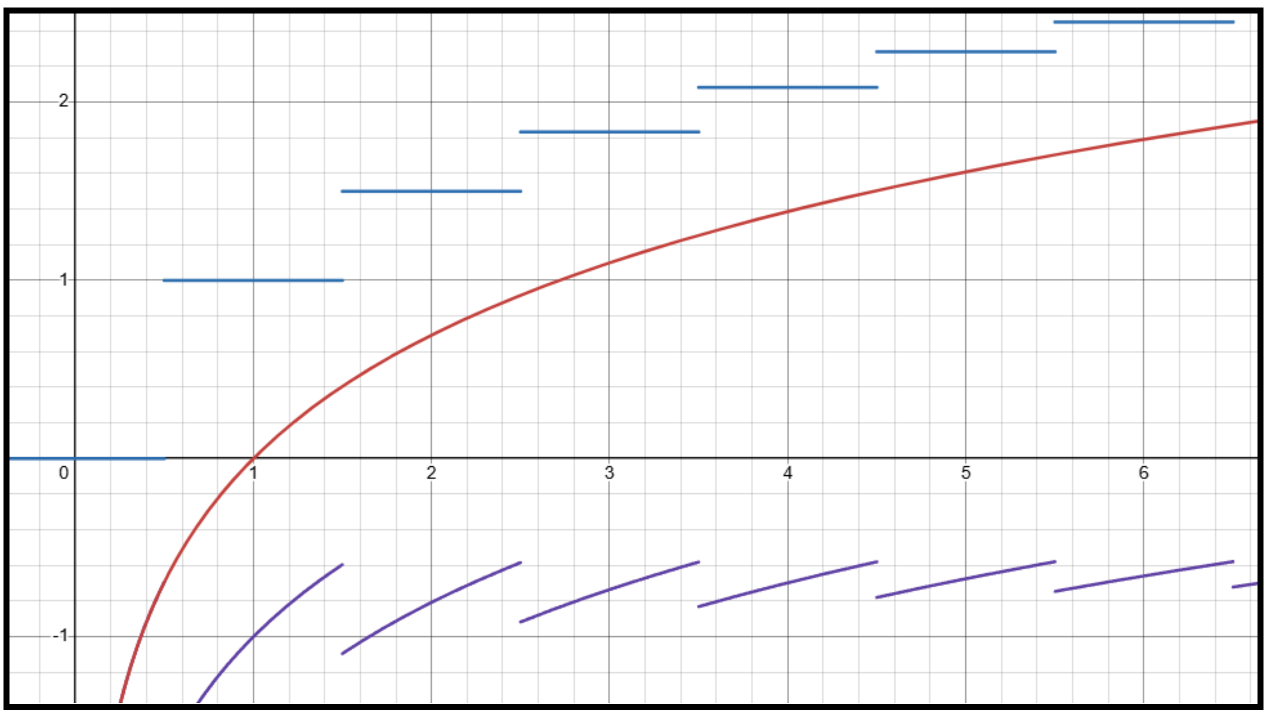
We can plot the following functions:

$$y = ln\, x$$

$$y = \sum_{n=1}^{x} \frac{1}{n}$$

$$y = ln\, x - \sum_{n=1}^{x} \frac{1}{n}$$

The result is the following graph:



The natural log is displayed in red. The summation function is displayed in blue. The difference is displayed in purple. Image by author. Powered by Desmos.

The reader will notice that the right edge of the difference tends towards γ_e corresponding to some $n + 0.5$. This allows us to write a program to tease out only these leading-edge values for any n. I include a Python version of the program here.

```
### ESTIMATION OF GAMMA ETA
import math

### Initiate Values
harmonic = 0
ln = 0
gamma_eta = 0
go = 1

while go == True:

    n = int(input("To what value of n the gamma? "))

    if n == 0:   ### End program condition
        go = 0
        print ("End of line.")
        break

    ### Calculate Values
    harmonic = 0
    for i in range(1, n + 1):
        harmonic = harmonic + 1 / i
    ln = math.log(n + 0.5)
    gamma_eta = ln - harmonic

    ### Print result in columns up to 10 decimal places
    print(n, "\t", f"{harmonic:.10}", "\t", f"{ln:.10}",
          "\t", f"{gamma_eta:.10}")
```

I include sample outputs from the program:

```
To what value of n the gamma? 10
10    2.928968254    2.351375257    -0.5775929968
To what value of n the gamma? 20
20    3.597739657    3.020424886    -0.577314771
To what value of n the gamma? 100
100   5.187377518    4.610157727    -0.5772197901
To what value of n the gamma? 200
200   5.878030948    5.300814247    -0.5772167014
To what value of n the gamma? 300
300   6.28266388     5.705447754    -0.5772161263
To what value of n the gamma? 1000
1000  7.485470861    6.908255154    -0.5772157065
```

The reader will observe the result that as n approaches infinity, the limit approaches the Euler gamma constant.

A CASE FOR THE LOGARITHMS OF NEGATIVE ARGUMENTS AS "DEFINED"

In orthodox practice, the logarithms of negative numbers are "undefined." This is due to the fact that the negative arguments lead to impossible contradictions. Yet here I will argue that negative arguments are possible. If we allow the natural logarithm to be defined as

$$ln(z) = \sum_{n=1}^{z-1} \frac{1}{n} + \frac{1}{2z} + \frac{1}{z^2 \cdot 12 + 1} - \gamma_e$$

then, analyzing each component, it is quite clear that each component can have a negative value. Clearly, the second, third, and fourth terms can have negative values. The sum may have a negative value as well—for example, as if summing backwards on the number line by using negative values for n and z—but this would equate to simply completing the sum in the usual way and multiplying the result by negative one. So then, the result would be

$$ln(-z) = -\sum_{n=1}^{z-1} \frac{1}{n} - \frac{1}{2z} - \frac{1}{z^2 \cdot 12+1} + \gamma_e$$

To examine what this might mean visually, we can investigate the Euler identity:

$$e^{i\pi} = -1$$

We use this identity to understand logs:

$$ln\left(e^{i\pi}\right) = ln\,(-1)$$

$$i\pi = ln\,(-1)$$

Generalizing for any value x:

$$ln\,(-1) = i\pi$$

$$ln\,(x) + ln\,(-1) = ln\,(x) + i\pi$$

$$ln\,(-x) = ln\,(x) + i\pi$$

What remains is to interpret the meaning of $i\pi$ in the final equation. It is understood that i signifies rotation and that π represents half of a complete rotation; therefore, $i\pi$ is equivalent to a flip. This is supported by the idea that

$$ln\,(-1) = i\pi$$

$$ln\left(i^2\right) = i\pi$$

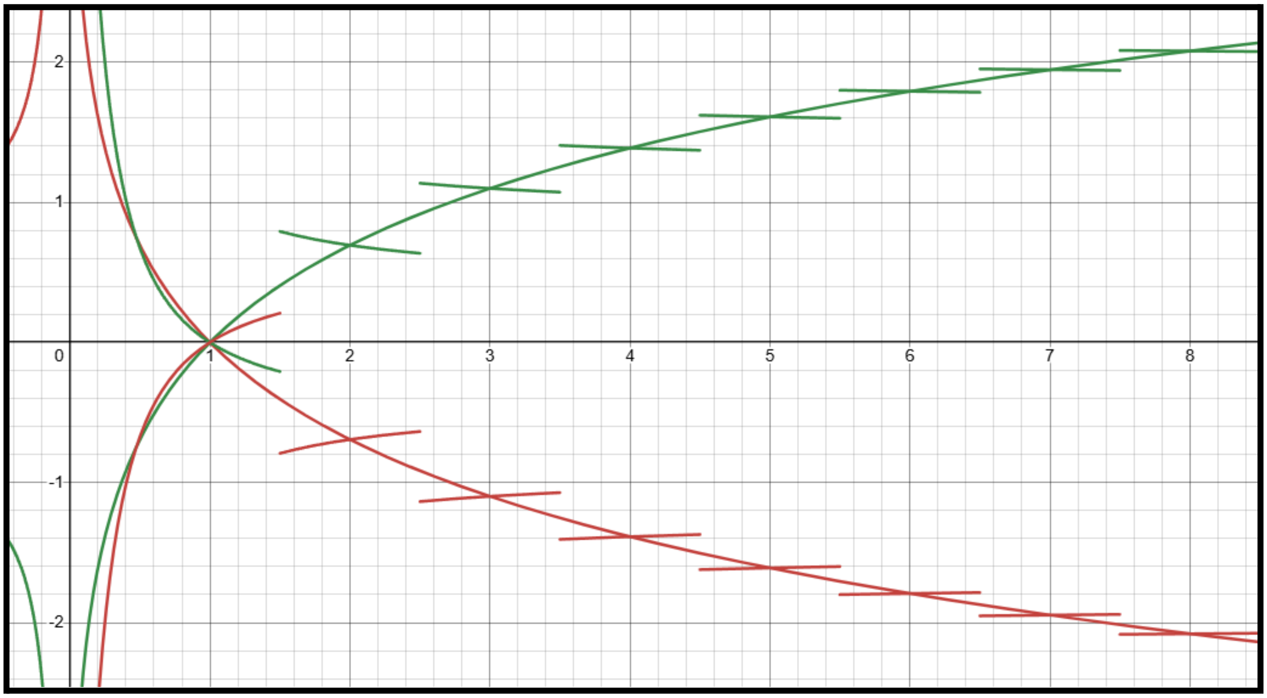
$$2\,ln\,i = i\pi$$

$$ln\,i = i\frac{\pi}{2}$$

where the latter equation represents a 90 degree rotation. So if a 180 degree rotation represents a flip, it remains to ask whether

$$ln\,(-x) = -\,ln(x)$$

indicates a vertical or horizontal flip. If we resort to the initial argument—that of multiplying the four terms of the Φ approximation by negative one—then a vertical flip seems to be the result as imaged in this graph.



The Φ approximation and the natural log are displayed in green; their negative counterparts in red. Image by author. Powered by Desmos.

The astute mathematician may argue that this is merely a special case of complex logarithms, and it is difficult to argue against that. In any case, the result is that negative numbers are replaced with rotations such that the contradictions inherent in negative logarithmic arguments are removed.

EULER'S e AS THE SURFACE OF A SPHERE

We have observed above the following:

$$e^{i\pi} = -1$$

$$ln\left(e^{i\pi}\right) = ln\,(-1)$$

$$ln\,(x) + ln\,(-1) = ln\,(x) + i\pi$$

$$ln\,(-x) = ln\,(x) + i\pi$$

Notice that the final equation can be expressed in two dimensions as shown on the graph. So then, what is the dimensionality of the Euler Identity? Generally, we find that taking the natural log of an expression reduces dimensionality because it produces a numeric result. Yet if the reduced dimension is two, then I will posit that the original dimension is three.

Let us first be reminded from where Euler's Identity is derived. In 1712 Roger Cotes, in his attempt to find the surface area of an ellipsoid, found in the relationship of two of his equations, that

$$ln\,(cos\,\phi\ +\ i\,sin\,\phi)\ =\ i\phi$$

Euler's Formula can be derived directly from this equation in two steps:

$$e^{ln\,(cos\,\phi\,+\,i\,sin\,\phi)} = e^{i\phi} \quad\rightarrow\quad cos\,\phi\ +\ i\,sin\,\phi = e^{i\phi}$$

Euler, seemingly unaware of Cote's result, published the formula in 1748 in the form,

$$e^{ix}\ =\ cos\,x\ +\ i\,sin\,x$$

which he derived by arranging series for e, sine and cosine (Wilson, 2017, p.12).

Here I use theta as it is commonly used for angles in education.

$$e^{i\theta}\ =\ cos\,\theta\ +\ i\,sin\,\theta$$

A special case occurs when $\theta\ =\ \pi$:

$$e^{i\pi}\ =\ cos\,\pi\ +\ i\,sin\,\pi\ =\ 1\ +\ 0\ =\ 1$$

$$e^{i\pi}\ -\ 1\ =\ 0$$

This is the famous Euler's Identity.

I will now demonstrate how this formula represents the surface of a sphere. Consider a three-dimensional reference frame where i is vertical, x is horizontal, and y is depth, where $+\ y$ is "out of the page" and $-\ y$ is "into the page." Imagine the radius as a vector originating from the origin. Finally, consider the postulation that

$$cos\,(i\theta)\ =\ i\,cos\,\theta$$

The logic is: Since i is a rotation, the argument $(i\theta)$ represents a 90 degree rotation, and this does not vary inside or outside of the trigonometric argument.

Using this postulation, a chart can be built with various angles. The value **r** is included as a proof of a spherical space, since the radius **r** of a unit sphere must always equal 1. The value 2π is used in place of $\theta\ =\ 0$.

$$e^{i\theta} = cos\,\theta + i\,sin\,\theta\,;\,r = \sqrt{\left|a^2 + b^2\right|}$$

θ	RESULT	r	θ	RESULT	r
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	$\frac{\pi}{2}$	$0 + i$	1
$\frac{\pi}{4}i$	$\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	$\frac{\pi}{2}i$	$0i - 1$	1
$-\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	$-\frac{\pi}{2}$	$-0 - i$	1
$-\frac{\pi}{4}i$	$\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$-\frac{\pi}{2}i$	$-0i + 1$	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	π	$-1 + 0i$	1
$\frac{3\pi}{4}i$	$-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	πi	$-i - 0$	1
$-\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	$-\pi$	$-1 - 0i$	1
$-\frac{3\pi}{4}i$	$-\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$-\pi i$	$i + 0$	1
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	$\frac{3\pi}{2}$	$0 - i$	1
$\frac{5\pi}{4}i$	$-\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$\frac{3\pi}{2}i$	$0i + 1$	1
$-\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	$-\frac{3\pi}{2}$	$0 + i$	1
$-\frac{5\pi}{4}i$	$-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	$-\frac{3\pi}{2}i$	$-0i - i$	1
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	2π	$1 + 0i$	1
$\frac{7\pi}{4}i$	$\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$2\pi i$	$i - 0$	1
$-\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	-2π	$1 - 0i$	1
$-\frac{7\pi}{4}i$	$\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	$-2\pi i$	$-i + 0i$	1

Let's look at how the math works beginning with the $\frac{\pi}{2}$ variants. First, $\frac{\pi}{2}$ results in i, and this is correct as it is the standard y-position—that is, the vertical "top" of the unit circle; $\frac{\pi}{2}i$ results in -1 for y, indicating a rotation from the "top" that circles "into the page"; $-\frac{\pi}{2}$ results in $-i$, the vertical "bottom" of the circle; and $-\frac{\pi}{2}i$ results in 1 for y, a rotation from the "bottom out from the page" or from the "top" circling contrariwise "out of the page"—it all depends on if the reader wants to treat the $\frac{\pi}{2}$ or the i as negative. To observe the rotations from cosine, we can examine the $\theta = \pi$ variants. The result of π is -1 for x, which is correct; for πi, the result is $-i$, indicating a counterclockwise rotation; for $-\pi$, the result is -1 for x, which is correct; and for $-\pi i$, the result is i, indicating a clockwise rotation from -1. I will now reduce the table by eliminating negative angles.

$e^{i\theta} = cos\,\theta + i\,sin\,\theta\,; r = \sqrt{\lvert a^2 + b^2 \rvert}$					
θ	**RESULT**	**r**	θ	**RESULT**	**r**
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	$\frac{\pi}{2}$	$0 + i$	1
$\frac{\pi}{4}i$	$\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	$\frac{\pi}{2}i$	$0i - 1$	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$	1	π	$-1 + 0i$	1
$\frac{3\pi}{4}i$	$-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}$	1	πi	$-i - 0$	1
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	$\frac{3\pi}{2}$	$0 - i$	1
$\frac{5\pi}{4}i$	$-\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$\frac{3\pi}{2}i$	$0i + 1$	1
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	2π	$1 + 0i$	1
$\frac{7\pi}{4}i$	$\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$	1	$2\pi i$	$i - 0$	1

This table is helpful because it contains only the 14 points of interest (six points representing the top, bottom, front, back, and side points, and two planar sets of four points between each of those pairs). Two redundancies occur as i and $-i$ result in both sine and cosine rotations. For $\theta = \frac{\pi}{4}$, the x position is $\frac{\sqrt{2}}{2}$ "to the right" and the i position is $\frac{\sqrt{2}}{2}i$ "up," which is correct; $\frac{\pi}{4}i$ results in $\frac{\sqrt{2}}{2}i$ (x rotates "up") and $-\frac{\sqrt{2}}{2}$ "back" into the y-axis. And so forth.

I apologize for the poor wording of this section. I understand that, mathematically, examples do not constitute a proof. But I do believe these examples demonstrate the essential points on the surface of a sphere.

THE OMEGA CONSTANT

The Omega Constant is defined by the equation

$$\Omega e^{\Omega} = 1$$

Most often it is considered an instance of the Lambert Function

$$W(z) = we^{w} = z$$

for $z = 1$. This discussion will approach the Omega Constant differently.

We begin by asking the question, "For what number is the square root equal to its natural log?" The question can be expressed this way:

$$\sqrt{x} = ln\, x$$

An equivalent and perhaps more convenient expression:

$$x = (ln\, x)^{2}$$

And if we want to generalize the question for any root, we can write:

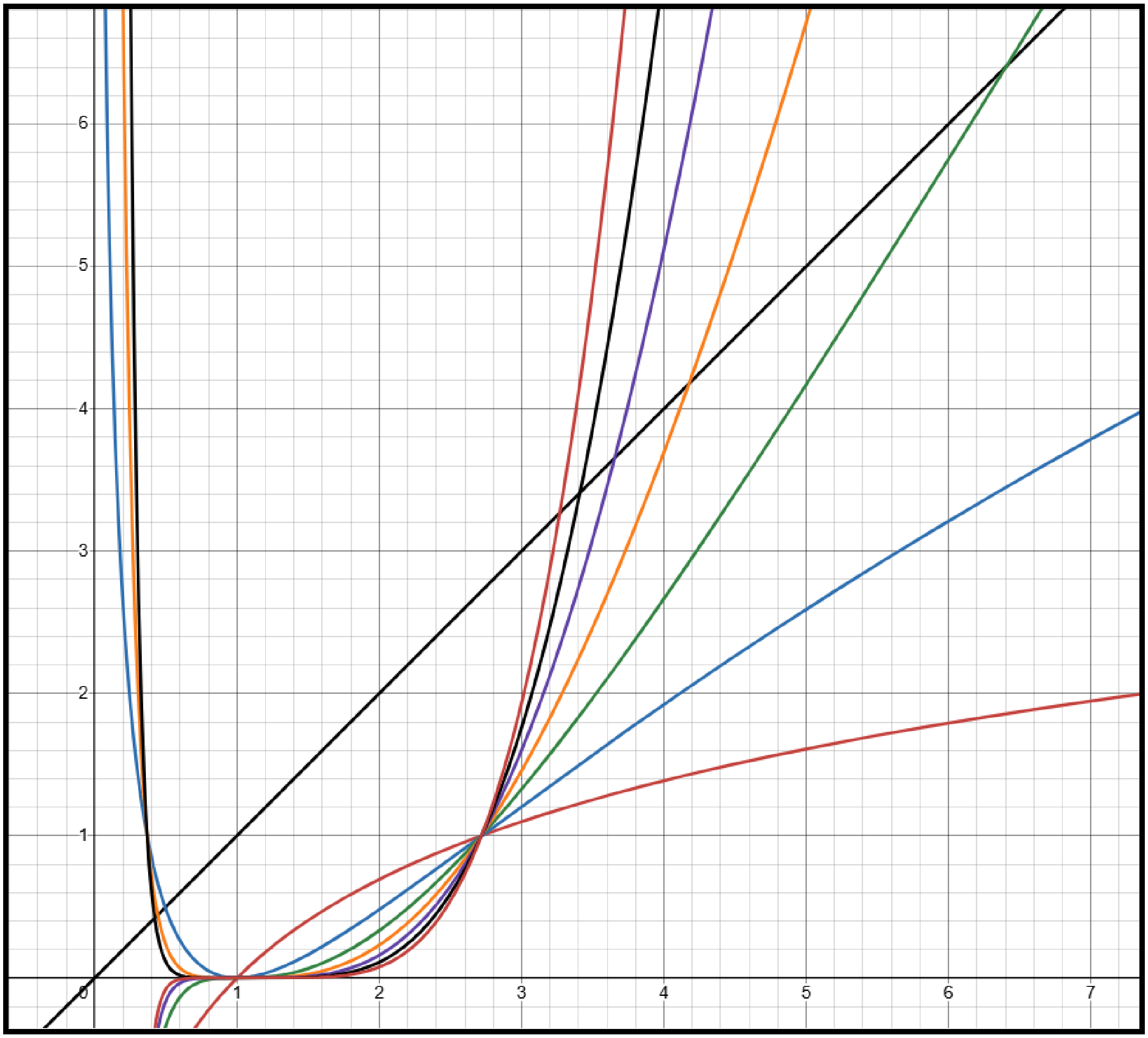
$$x = (ln\, x)^{R}$$

To investigate the answer to the question using the Desmos graphing calculator, we employ the equation in the form $y = (ln\, x)^{R}$. We will also employ

$$y = x$$

The student can consider this equation as stating, "What is on the left side of the equation is equal to what is on the right side of the equation." Therefore, any line or curve that crosses this slope $m = 1$ line fulfills that requirement. (I speak as a high school teacher.) So let's graph a number of these equations and see where the cross the $y = x$ line. The following image graphs the equations:

$y = x$
$y = ln\, x$
$y = (ln\, x)^2$
$y = (ln\, x)^3$
$y = (ln\, x)^4$
$y = (ln\, x)^5$
$y = (ln\, x)^6$
$y = (ln\, x)^7$



The graph of the equations $y = (ln\, x)^R$ *for* $R = \{1, 2, 3, 4, 5, 6, 7\}$ *and the equation* $y = x$*. Image by author. Powered by Desmos.*

The reader has already noticed a few matters of interest: that only $y = ln\, x$ [lower red] does not satisfy the condition (i.e. does not cross the $m = 1$ line), that all other odd values of R satisfy the condition once, that $y = (ln\, x)^2$ [blue] satisfies the condition once, and that all other even values of R satisfy the condition twice. I include a table of values for investigation.

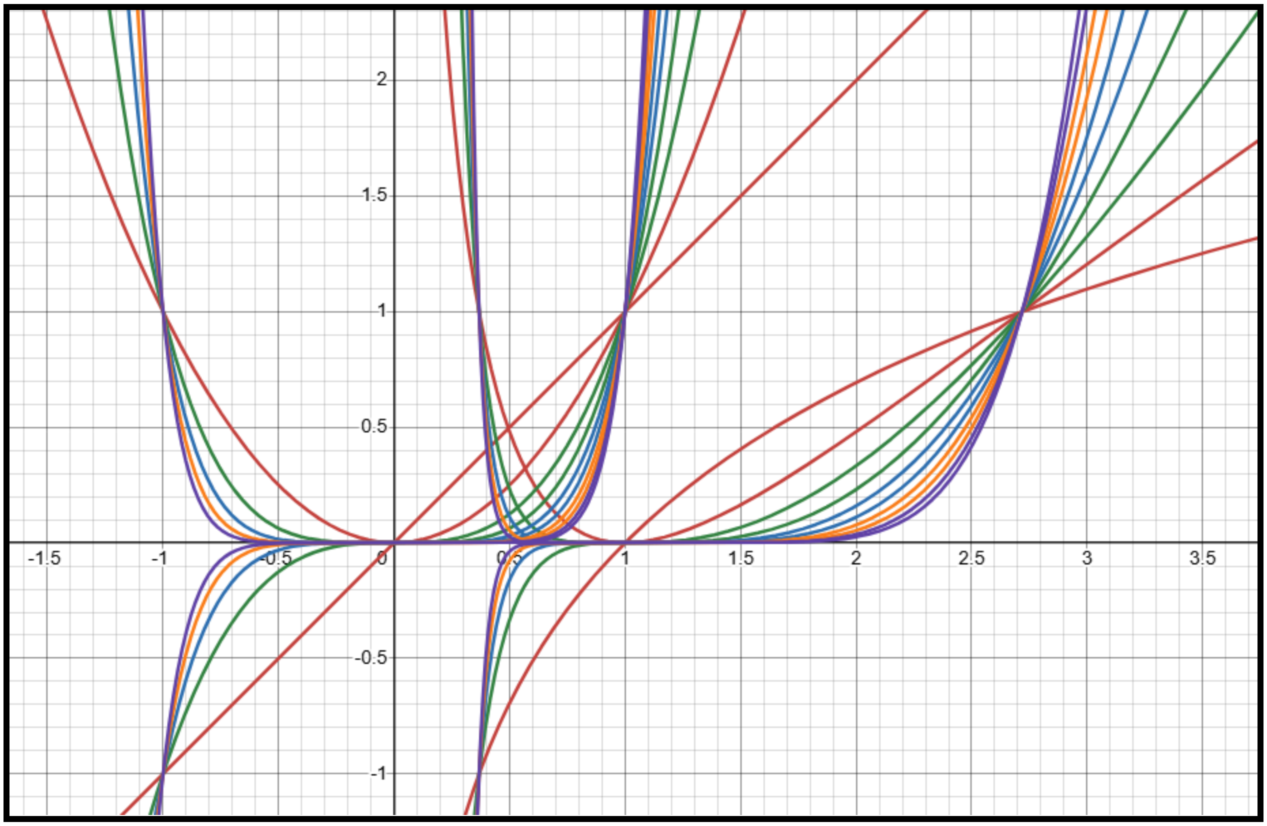
Points Satisfying $y = x$		
EQUATION	$VALUE < 1$	$VALUE > 1$
$y = (ln\, x)^1$	∅	∅
$y = (ln\, x)^2$	0.494866	∅
$y = (ln\, x)^3$	∅	6.40567
$y = (ln\, x)^4$	0.442394	4.17708
$y = (ln\, x)^5$	∅	3.65412
$y = (ln\, x)^6$	0.420778	3.41060
$y = (ln\, x)^7$	∅	3.26856

Our next step will be to graph the associated curve (or line in one instance). For example, for $y = (ln\, x)^1$ we will graph the line $y = x$, for $y = (ln\, x)^2$ we will graph the curve $y = x^2$, for $y = (ln\, x)^3$ we will graph the curve $y = x^3$, and so forth. Doing so will allow us to look for a relationship between the pairs of equations, which can be written as

$$y = (ln\, x)^R$$

$$y = x^R$$

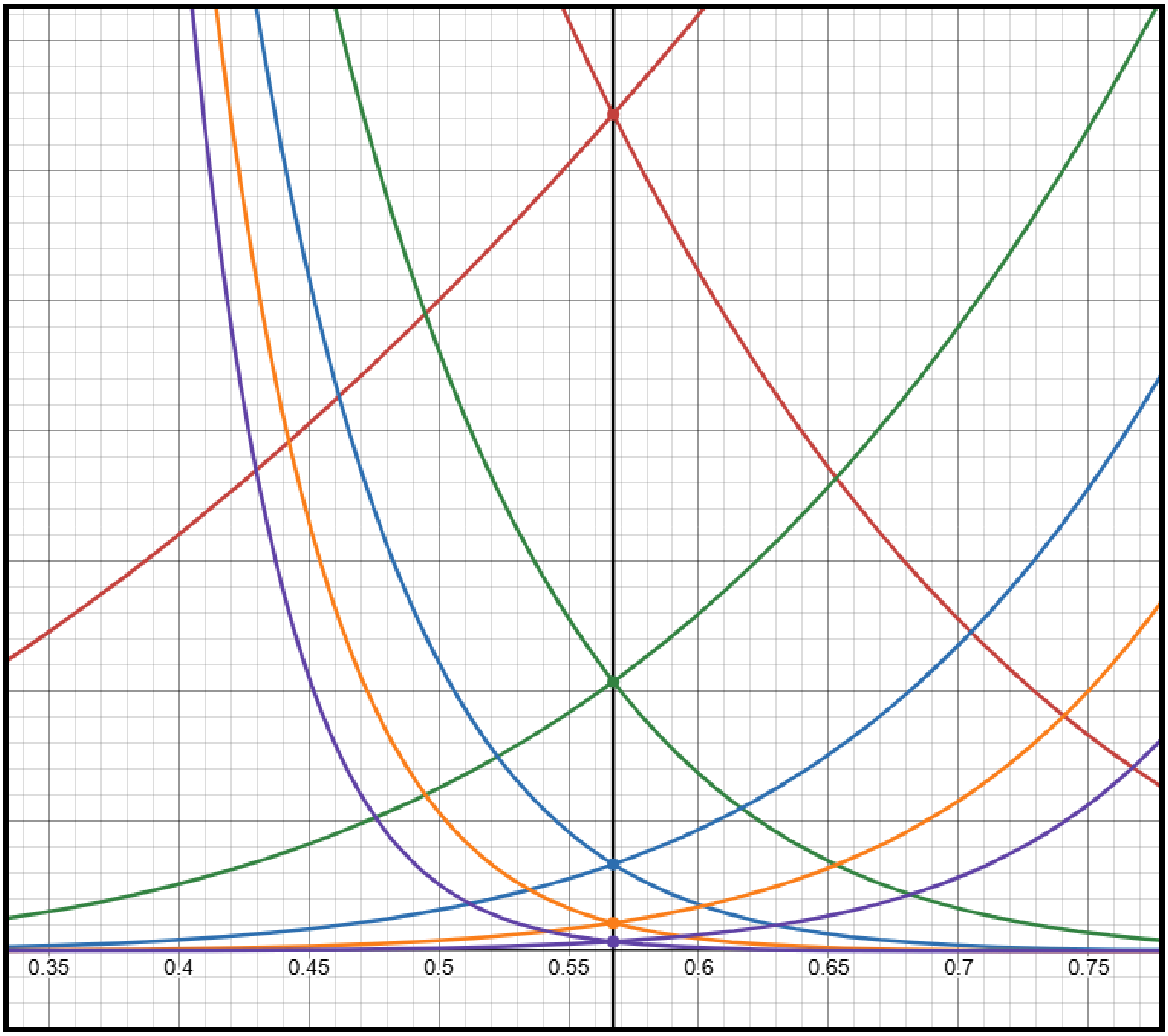
We can also look for patterns in the odd-value R equations, the even-value R equations, and the full set of equations.



The full set of all equations. R = 1, 2 are in red, R = 3, 4 are in green, R = 5, 6 are in blue, R = 7, 8 are in orange, and R = 9, 10 are in purple. Image by author. Powered by Desmos.

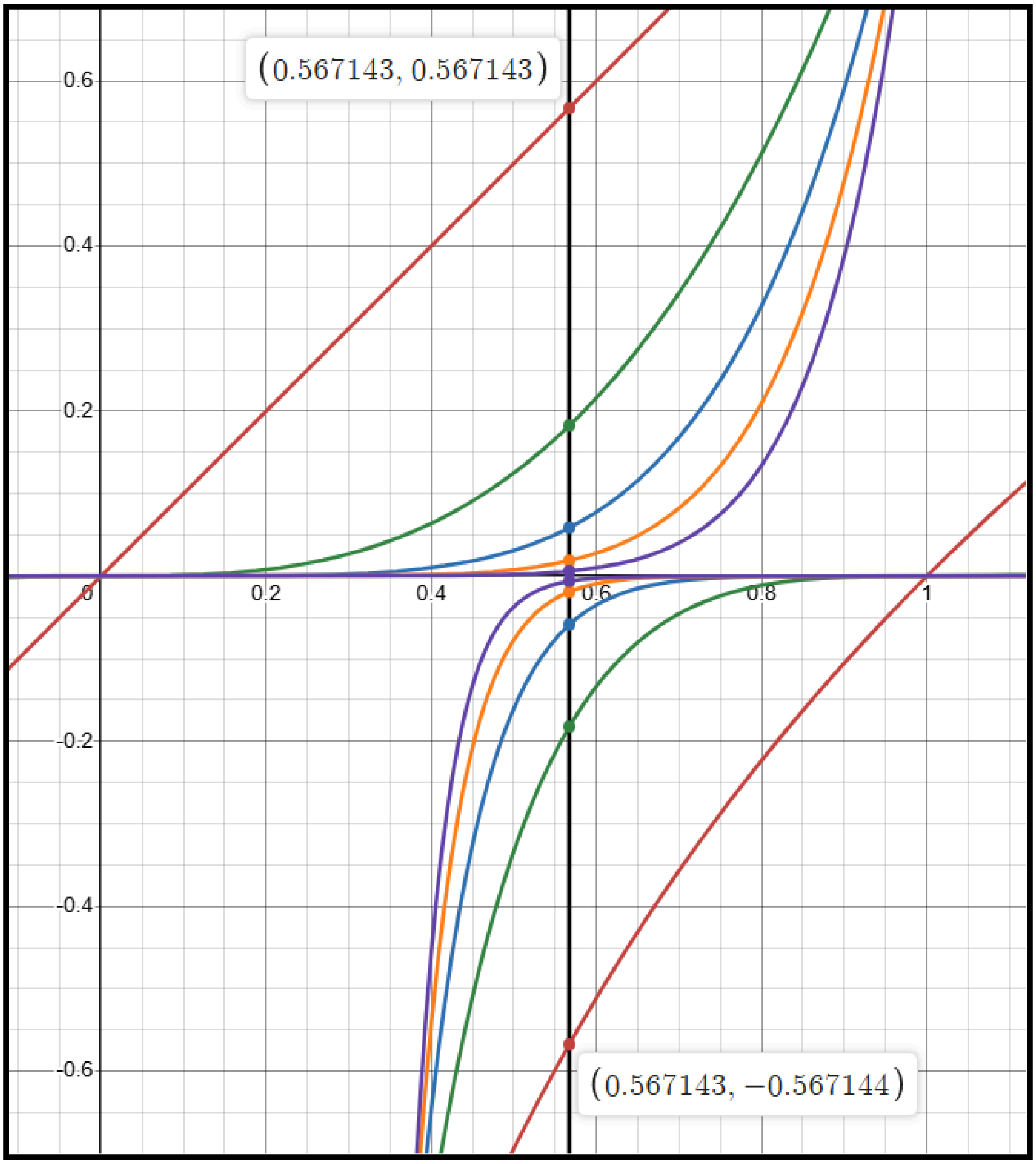
A number of points present themselves rather unsurprisingly. These are $(-1, 1)$, $(-1, -1)$, $(1, 1)$, $(e, 1)$, $\left(\frac{1}{e}, 1\right)$ and $\left(\frac{1}{e}, -1\right)$.

We will presently investigate the Omega Constant. This constant presents itself in both the odd-value R equations and the even-value R equations, but does so differently. Let us examine the even exponent equations first as the relationship is most easily observed.



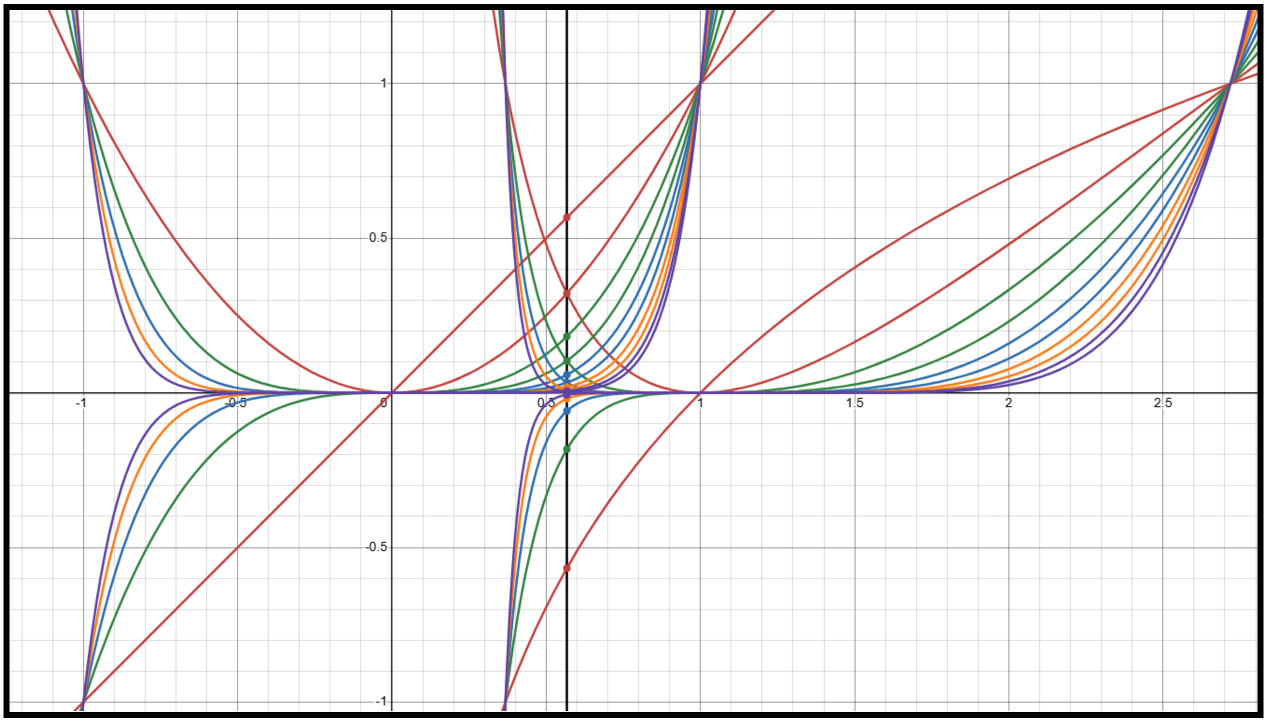
The equations $y = x^R$ *and* $y = (\ln x)^R$ *for* $R = \{2, 4, 6, 8, 10\}$*. The vertical line (black) is the Omega Constant* $x = \Omega = 0.567143$*. Image by author. Powered by Desmos.*

For all even exponential values of R, including $R = 2$, the associated equations connect at $x = \Omega$. The y-value begins at $y = 0.321651$ for $R = 2$ and continues to decrease as R increases.



The equations $y = x^R$ *and* $y = (\ln x)^R$ *for* $R = \{1, 3, 5, 7, 9\}$. *The vertical line (black) is the Omega Constant* $x = \Omega = 0.567143$. *Image by author. Powered by Desmos.*

In the case of the odd values for R, let the reader notice that the equations are evenly spaced on the vertical line $x = \Omega$. For example, for $R = 9$ (the purple curves), the two points of interest are $(0.567143,\ 0.006071)$ and $(0.567143,\ -0.006071)$. As with the even values of R, the line $x = \Omega$ behaves as a local axis of symmetry. Remarkable!



The full set of all equations. R = 1, 2 are in red, R = 3, 4 are in green, R = 5, 6 are in blue, R = 7, 8 are in orange, and R = 9, 10 are in purple. This image includes the vertical line $x = \Omega$ and marks the critical points. Image by author. Powered by Desmos.

Analyses of the images require us to ask the question, “Do the Omega Points occur where the slopes of the associated curves are equal or negative inverses?” We can easily investigate this.

$$y_1 = (ln\, x)^R \quad \rightarrow \quad y_1' = \frac{R\,(ln\, x)^{R-1}}{x}$$

$$y_2 = x^R \quad \rightarrow \quad y_2' = Rx^{R-1}$$

But, wait a minute! There's an easier way! Using the Online Arbitrary Precision Calculator (2022), we observe that

$$ln\,(0.56714329040978) = -0.56714329040978$$

This means that at the point $x = \Omega$, we have the equations and derivatives:

$$y_1 = (-\Omega)^R \quad \rightarrow \quad y_1' = R(-\Omega)^{R-1}$$

$$y_2 = \Omega^R \quad \rightarrow \quad y_2' = R(\Omega)^{R-1}$$

Therefore, at the point $x = \Omega$, the values of the functions y_1 and y_2 will be equal if R is even, but the slopes will be the negative inverse of the other (because the exponent of derivative is odd); conversely, at the point $x = \Omega$, the values of the functions y_1 and y_2 will be the negative inverses of each other if R is odd, but the slopes will be equal (because the exponent of derivative is even). And this is what we see in the graph.

SPECIAL THANKS

Thank you to Christian Williams for proofreading and verifying this paper. Mahalo!

COMMENT

I believe the discoveries of this paper to be original, and, if not, they are "original to myself" and I apologize to those who arrived here first. Mahalo!

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