MATHEMATICA EXPLORATIO I

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With great joy I present this paper to you. I would not waste your time, so I say that I do believe that this paper includes what is original and useful. But if the contents is not original, then please accept my apologies. I am not a professional mathematician, but a high school teacher with a BA minor in mathematics. I can only say that the theorems presented here are original to me. Please also forgive my writing: Since I am not a professional mathematician, the theorems, ideas, and other content are in the language of a high school teacher instructing students.

But back to my first point. I hope this paper will be useful to the student and perhaps even to the teacher. God bless!

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SECTION A

PARABOLAS

Two Alternatives to the Quadratic Formula

SOLVING IN 3-SPACE

$$az^2 = ax^2 + bx + c$$

DEFINITIONS

$y = ax^2 + bx + c$	Standard form of a vertical parabola.
x_m	The x-value of a parabola's line of symmetry; the x-value where the parabola has either its maximum or minimum y-value.
y_m	The maximum or minimum y-value of the parabola.
$V(x_m, y_m)$	The vertex of the parabola.
(x, y, z)	The 3-space dimension, equivalent to the 2-space with imaginary $(x + zi, y)$.
x_0	The "roots" of an equation; the x-values of an equation where $y = 0$; the solutions to the quadratic formula.

3-Space Method

Consider the parabola

$$y = ax^2 + bx + c$$

Complete the following steps:

	$y = ax^2 + bx + c$
Replace y with az^2.	$az^2 = ax^2 + bx + c$
Find x_m.	$x_m = -\frac{b}{2a}$
Place the x_m into the equation.	$az^2 = a{x_m}^2 + bx_m + c = y_m$
You now have the vertex.	(x_m, y_m)
Solve for z.	$z = \sqrt{\frac{y_m}{a}}$
If z is imaginary, the solutions are:	$x_0 = x_m \pm z$
If z is real, the solutions are:	$x_0 = x_m \pm zi$ or $(x_m, 0, \pm z)$

Example with Solution in 2-Space

	$y = -5x^2 + 7x + 2$
Replace y with az^2.	$-5z^2 = -5x^2 + 7x + 2$
Find x_m.	$x_m = -\frac{b}{2a} = \frac{7}{10} = 0.7$
Place x_m into the equation and solve.	$-5z^2 = -5(0.7)^2 + 7(0.7) + 2 = 4.45$
You now have the vertex.	$V(0.7, 4.45)$
Solve for z.	$z = \sqrt{\frac{4.45}{-5}} = 0.943i$
If z is imaginary, the solutions are:	$x_0 = 0.7 \pm 0.943 = 1.643\ and - 0.243$
If z is real, the solutions are:	z is not real

Example with Solution in 3-Space

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Place x_m into the equation and solve.	$az^2 = 10(-0.25)^2 + 5(-0.25) + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

3-Space Method with y_m Shortcuts

$$y_m = -a{x_m}^2 + c$$

$$y_m = \frac{b}{2}x_m + c$$

Applying the first y_m shortcut to the 3-Space Method

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Use first y_m shortcut.	$az^2 = -10(-0.25)^2 + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

Applying the second y_m shortcut to the 3-Space Method

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Use second y_m shortcut.	$az^2 = \frac{5}{2}(-0.25) + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

PROOF OF 3-SPACE METHOD

$$y = ax^2 + bx + c - az^2$$

The equation above is constructed as a 3-space equivalent of a parabola given by:

$$y = ax^2 + bx + c$$

Finding roots for a parabola occurs by definition when $y = 0$. The proof proceeds with the following steps:

	$y = ax^2 + bx + c$
Replace 2-Space equation with 3-Space equation.	$y = ax^2 + bx + c - az^2$
Set $y = 0$.	$0 = ax^2 + bx + c - az^2$
Move az^2 to the left side of the equation.	$az^2 = ax^2 + bx + c$
Solve for z.	The z-roots are found by definition at $y = 0$ and $x = x_m$.

PROOFS OF y_m SHORTCUTS

Evidence of these relationships become apparent to any teacher who has taught Algebra II long enough. The proofs are helpful.

$$y_m = -a{x_m}^2 + c$$

	$y = ax^2 + bx + c$
Let $x = x_m = -\frac{b}{2a}$.	$y_m = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$
Apply and reduce.	$y_m = \frac{b^2}{4a} - \frac{b^2}{2a} + c$
Let $Q = \frac{b^2}{2a}$.	$y_m = \frac{Q}{2} - Q + c$
Solve.	$y_m = -\frac{Q}{2} + c$
Replace $Q = \frac{b^2}{2a}$.	$y_m = -\frac{b^2}{4a} + c$
Generate ${x_m}^2$.	$y_m = \frac{b^2}{4a^2} \cdot K + c$
Solve for K to maintain equivalent.	$y_m = \frac{b^2}{4a^2} \cdot (-a) + c$
Shortcut 1 is proven.	$y_m = {x_m}^2 \cdot (-a) + c$

$$y_m = \frac{b}{2} x_m + c$$

	$y = ax^2 + bx + c$
Let $x = x_m = -\frac{b}{2a}$.	$y_m = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$
Apply and reduce.	$y_m = \frac{b^{\,2}}{4a} - \frac{b^2}{2a} + c$
Let $Q = \frac{b^2}{2a}$.	$y_m = \frac{Q}{2} - Q + c$
Solve.	$y_m = -\frac{Q}{2} + c$
Replace $Q = \frac{b^2}{2a}$.	$y_m = -\frac{b^2}{4a} + c$
Extract $x_m = -\frac{b}{2a}$.	$y_m = -\frac{b}{2a} \cdot \frac{b}{2} + c$
Shortcut 2 is proven.	$y_m = x_m \cdot \frac{b}{2} + c$

THE X-MAX METHOD

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

DEFINITIONS

$y = ax^2 + bx + c$	Standard form of a vertical parabola.
x_m	The x-value of a parabola's line of symmetry; the x-value where the parabola has either its maximum or minimum y-value.
x_0	The "roots" of an equation; the x-values of an equation where $y = 0$; the solutions to the quadratic formula.

This method, when performed by hand, is often simpler than the Quadratic Formula, but not always. However, with a calculator it is more straightforward and reduces the probability of errors.

X-Max Method

Consider the parabola

$$y = ax^2 + bx + c$$

Complete the following steps:

	$y = ax^2 + bx + c$
Find x_m.	$x_m = -\frac{b}{2a}$
Square x_m, subtract $\frac{c}{a}$, and take the square root.	$\left(x_m{}^2 - \frac{c}{a}\right)^{0.5}$
Solve by adding to or subtracting from x_m.	$x_0 = x_m \pm \left(x_m{}^2 - \frac{c}{a}\right)^{0.5}$

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

Example with Solution in 2-Space

	$y = -5x^2 + 7x + 2$
Find x_m.	$x_m = \frac{7}{10} = 0.7$
Using your calculator.	$0.7^2 + \frac{2}{5} = 0.89 \quad Ans^{.5} = 0.943$
Solve by adding to or subtracting from x_m.	$x_0 = 0.7 \pm 0.943 = 1.643\ and - 0.243$

Example with Solution in 3-Space

	$y = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{5}{20} = -0.25$
Using your calculator.	$.25^2 - \frac{4}{10} = -0.3375 \quad Ans^{.5} = 0.581i$
Solve by adding to or subtracting from x_m.	$x_0 = -0.25 \pm 0.581i$

PROOF OF X-MAX METHOD

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

The equation above is arrived at through tinkering with the Quadratic Formula.

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

	$x_0 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Break the Quadratic Formula into two terms.	$x_0 = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$
Square and root the denominator of the second term.	$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{\sqrt{2^2 a^2}}$
Combine the second term under the same radical.	$x = -\dfrac{b}{2a} \pm \sqrt{\dfrac{b^2}{2^2 a^2} - \dfrac{4ac}{2^2 a^2}}$
Simplify.	$x = -\dfrac{b}{2a} \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$
Substitute $x_m = -\frac{b}{2a}$.	$x = x_m \pm \sqrt{{x_m}^2 - \dfrac{c}{a}}$

Parabolic Point Equations

PARABOLA

$$P\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a}+c\right)$$

VERTEX

$$V\left(\frac{-b}{2a}, \frac{-b^2}{4a}+c\right)$$

FOCUS

$$F\left(\frac{-b}{2a}, \frac{1-b^2}{4a}+c\right)$$

DIRECTRIX POINT

$$D\left(\frac{-b}{2a}, \frac{-1-b^2}{4a}+c\right)$$

DEFINITIONS

m	The slope of a tangent line at any point of the parabola.
$Focus$	A point not on the parabola, but between two points of the parabola whose tangents have the slopes $m = 1$ and $m = -1$.
$Directrix\ Point$	A point not on the parabola, but between two points of the parabola whose tangents have the slopes $m = i$ and $m = -i$. The Directrix Point is opposite the Focus, equidistant from the Vertex.

$$P\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a} + c\right)$$

From the parabola point equation, which can be inferred empirically, the following slope equations are derived.

Ascertain the x equation.	$x = \dfrac{m-b}{2a}$
Multiply by $2a$.	$m - b = 2ax$
Solve for m.	$m = 2ax + b$

The calculus student will recognize the following relationship:

$y = ax^2 + bx + c$	$y' = 2ax + b$

Ascertain the y equation.	$y = \frac{m^2 - b^2}{4a} + c$
Adjust the second term.	$y = \frac{m^2 - b^2}{4a} + \frac{4ac}{4a}$
Combine.	$y = \frac{m^2 - b^2 + 4ac}{4a}$
Multiply by $4a$.	$4ay = m^2 - b^2 + 4ac$
Re-arrange to solve for m.	$m^2 = b^2 + 4ay - 4ac$
Extract $4a$.	$m^2 = b^2 + 4a(y - c)$
Solve for m.	$m = \sqrt{b^2 + 4a(y - c)}$

The algebra student will recognize the special case:

	$m = \sqrt{b^2 + 4a(y - c)}$
Let $y = 0$.	$m = \sqrt{b^2 - 4ac}$

Motion Applications

The Rocket Problem

$$v_* = m = \sqrt{b^2 - 4ac}$$

The rocket problem—perhaps applicable only to model rockets—can be expressed with the following variables:

$$H(t) = \frac{g}{2}t^2 + v_0 t + H_0$$

DEFINITIONS

$H(t)$	Height in meters as a function of time.
t	Time in seconds.
g	Gravity. As an approximation, $\frac{g}{2} = -5\frac{m}{s^2}$.
v_0	The initial velocity at liftoff.
H_0	The initial height; the height of the rocket platform.
v_*	Impact velocity.
t_*	Time of impact.
v_i	Instantaneous velocity.

For the rocket problem, $m = v_i = H'$. This is to say, the derivative of height is velocity, so the slope m is equivalent to velocity v_i.

$$v_i = \sqrt{(v_0)^2 + 4 \cdot \frac{g}{2} \cdot (H - H_0)}$$

$$H = 0 \rightarrow v_* = \sqrt{(v_0)^2 - 4 \cdot \frac{g}{2} \cdot H_0}$$

We will apply this to the Quadratic Formula. Since the impact time is positive, the second term must be positive (and both v_* and g have negative values).

$$x_0 = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \rightarrow t_* = -\frac{v_0}{g} + \frac{v_*}{g}$$

This allows for the following interpretations:

$$time\ of\ rocket\ impact = -\frac{initial\ velocity}{gravity} + \frac{impact\ velocity}{gravity}$$

$$time\ of\ rocket\ impact = time\ to\ climb + time\ to\ fall$$

$$t_* = -\frac{H'(0)}{H''} + \frac{H'(t_*)}{H''}$$

Applying the rocket problem to the parabola point equations presents additional interpretations. The Vertex Point Equation represents x_m and y_m analogously.

The Vertex Point Equation.	$V\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$
Substitute kinetic variables.	$V\left(\frac{v_0}{-g}, \frac{{v_0}^2}{-2g} + H_0\right)$
Apply integration of v_0.	$V\left(\frac{v_0}{-g}, \frac{\int v_0}{-g} + H_0\right)$

The Parabola Point Equation produces interesting results.

The Parabola Point Equation.	$P\left(\frac{m-b}{2a}, \frac{m^2 - b^2}{4a} + c\right)$
Substitute kinetic variables.	$P\left(\frac{v_i - v_0}{g}, \frac{{v_i}^2 - {v_0}^2}{2g} + H_0\right)$
Apply integration.	$P\left(\frac{v_i - v_0}{g}, \frac{\int v_i - \int v_0}{g} + H_0\right)$

The Newtonian Derivation

$$m = \sqrt{b^2 + 4a(y - c)}$$

The slope equation can be applied to any y-value and solved for any x-value. Thus, the Quadratic Formula, in a generalized form, will resolve into the Newtonian velocity equation.

Obtain the slope equation in terms of any y.	$m = \sqrt{b^2 + 4a(y - c)}$
Convert to kinetic terms for any instantaneous velocity v_i.	$v_i = m$
Generalize the Quadratic Formula for any time t and any acceleration A.	$t = -\frac{v_0}{A} + \frac{v_i}{A}$
Multiply by a.	$At = -v_0 + v_i$
Solve for instantaneous velocity.	$v_i = v_0 + At$

Parabola Theorems

Secant-Slope Theorem

$$m_s = \frac{m_1 + m_2}{2}$$

The slope m_s of a secant passing through two points of a parabola, where m_1 and m_2 are the slopes of the tangents at those points, is given by

$$m_s = \frac{m_1 + m_2}{2}$$

This holds whether the secant does or does not cross the axis of symmetry.

Using the parabola point equation, consider two points on a parabola distinguished by slopes m_1 and m_2.	$\left(\frac{m_1 - b}{2a}, \frac{{m_1}^2 - b^2}{4a} + c\right)$ $\left(\frac{m_2 - b}{2a}, \frac{{m_2}^2 - b^2}{4a} + c\right)$
Let the slope of the secant s be given by	$m_s = \frac{y_1 - y_2}{x_1 - x_2}$
Insert the x and y values from the parabola point equations into the equation for m_s.	$m_s = \frac{\left(\frac{{m_1}^2 - b^2}{4a} + c\right) - \left(\frac{{m_2}^2 - b^2}{4a} + c\right)}{\frac{m_1 - b}{2a} - \frac{m_2 - b}{2a}}$

Apply algebra.	$m_s = \dfrac{\dfrac{{m_1}^2 - {m_2}^2}{4a}}{\dfrac{m_1 - m_2}{2a}}$
Factor the upper numerator.	$m_s = \dfrac{\dfrac{(m_1 - m_2)(m_1 + m_2)}{4a}}{\dfrac{m_1 - m_2}{2a}}$
Multiply by the reciprocal of the denominator.	$m_s = \dfrac{m_1 + m_2}{2}$

Secant-Vertex Theorem

$$m_v = \frac{m_2}{2}$$

The slope m_v of a secant passing through the vertex and any other point on the parabola, where m_1 and m_2 are the slopes of the tangents at those points, is given by

$$m_v = \frac{m_2}{2}$$

Refer to the Secant-Slope Theorem	$m_s = \frac{m_1 + m_2}{2}$
The slope of the tangent passing through the vertex equals 0.	$m_1 = 0$
Apply the m_1 value to the Secant-Slope Equation.	$m_s = \frac{0 + m_2}{2}$
The theorem is proven.	$m_v = \frac{m_2}{2}$

Tangent Point Y-Value Theorem

$$l_y = -\frac{m^2}{4a}$$

Let m be the slope of the tangent of any point of a parabola except for the vertex. Let l_y represent the y-value distance between that point and the vertex. The length l_y will be given by the equation:

$$l_y = -\frac{m^2}{4a}$$

Refer to the vertex point equation.	$\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$
Refer to the parabola point equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a} + c\right)$
Let l_y equal the y-value of the vertex minus the y-value of the tangent point.	$l_y = \left(\frac{-b^2}{4a} + c\right) - \left(\frac{m^2-b^2}{4a} + c\right)$
Apply algebra.	$l_y = \left(\frac{-b^2}{4a}\right) - \left(\frac{m^2-b^2}{4a}\right)$
The theorem is proven.	$l_y = -\frac{m^2}{4a}$

Tangent Point X-Value Theorem

$$l_x = -\frac{m}{2a}$$

Let m be the slope of the tangent of any point of a parabola except for the vertex. Let l_x represent the x-value distance between that point and the vertex. The length l_x will be given by the equation:

$$l_x = -\frac{m}{2a}$$

Refer to the vertex point equation.	$\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$
Refer to the parabola point equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a} + c\right)$
Let l_x equal the x-value of the vertex minus the x-value of the tangent point.	$l_x = \left(\frac{-b}{2a}\right) - \left(\frac{m-b}{2a}\right)$
The theorem is proven.	$l_x = -\frac{m}{2a}$

The 2C Quadratic Formula

$$x_0 = 0 - \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

$$x_0 = -\frac{b}{a} + \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

This alternative to the Quadratic Formula may lack utility for the student as it is every bit as complex; however, one never knows when or for whom something may have utility, so I include it here.

The 2C Quadratic Formula operates differently than the standard Quadratic Formula. Rather than starting at the axis of symmetry and moving outward to the roots, the 2C Formula begins at 0 and $-\frac{b}{a}$ and moves outward to the roots.

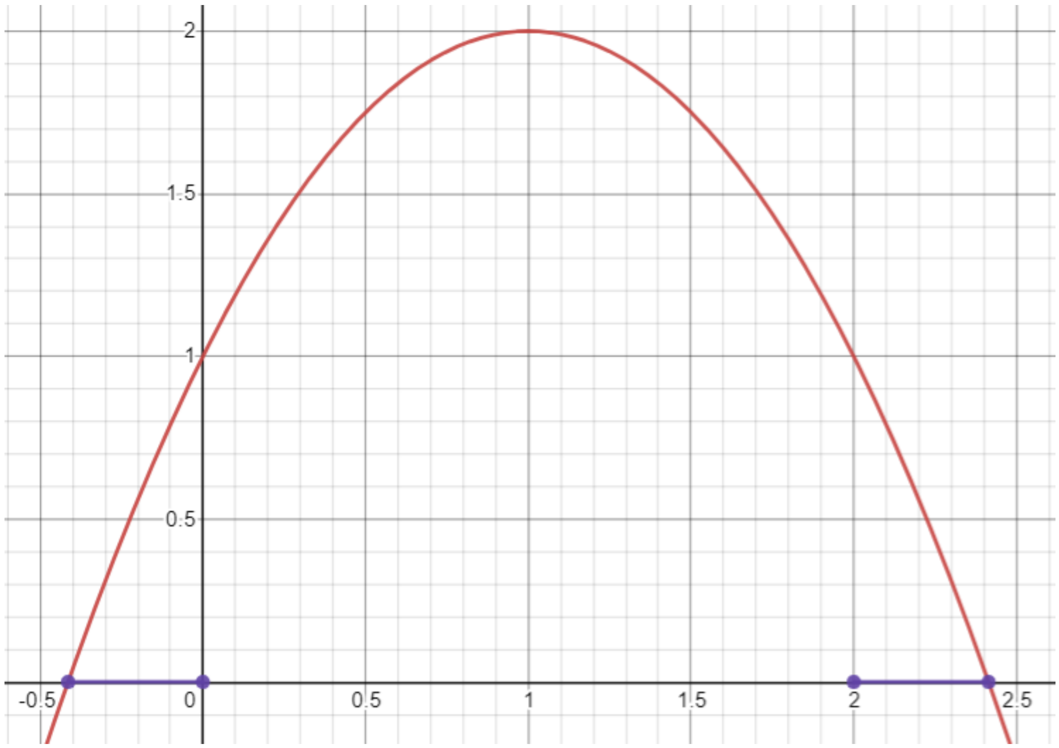
For convenience, the second term of the 2C Formula will be defined as k, giving

$$x_0 = 0 - k$$

$$x_0 = -\frac{b}{a} + k$$

Thus, to prove the 2C Formula, all that is necessary is to show that

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$



The distance k shown in purple. Image: Desmos

Proof of K-Value by Slopes

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

The value k can be derived using the Secant-Slope Theorem and the ratio of $\frac{c}{k}$. The Secant-Slope Theorem will provide the slope of the secant (black), which must be equal to the ratio of c (green) over k (blue). The only unknown is k.

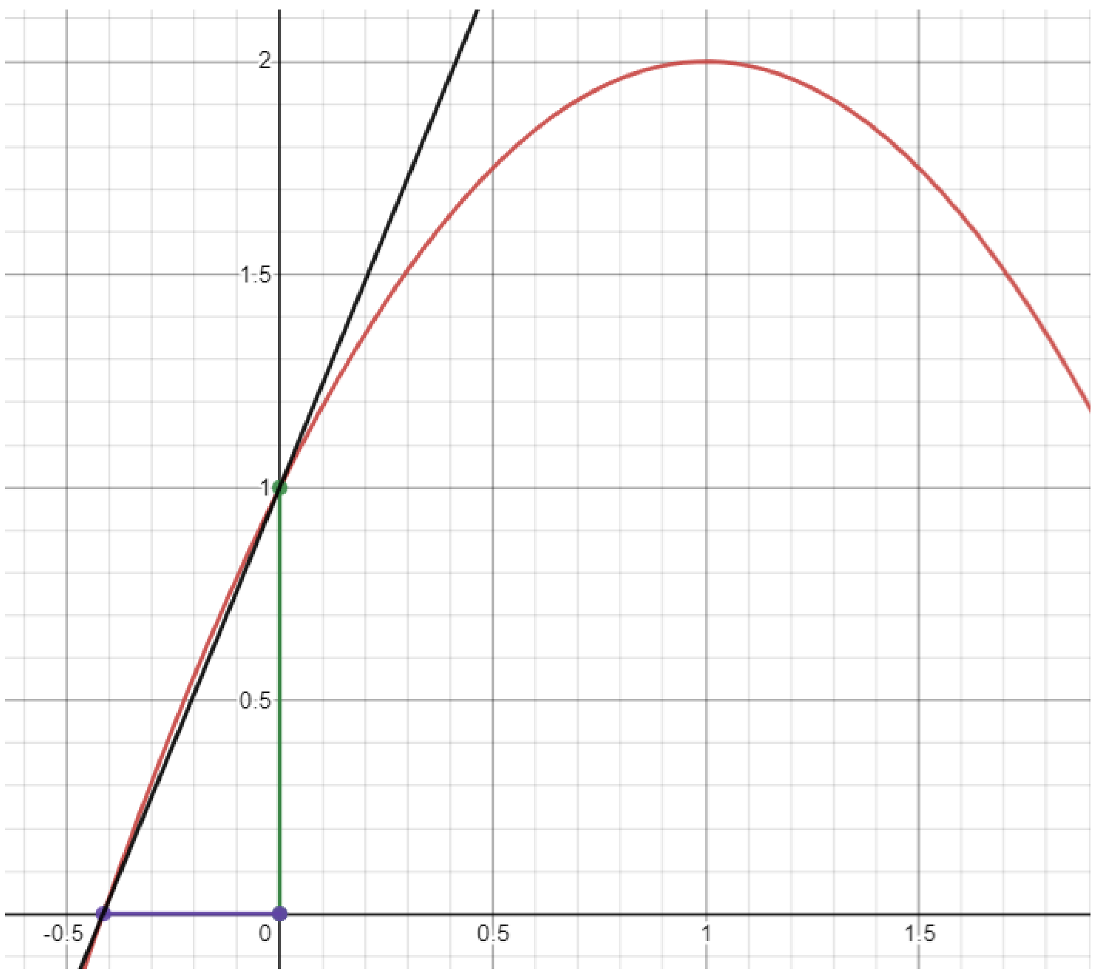


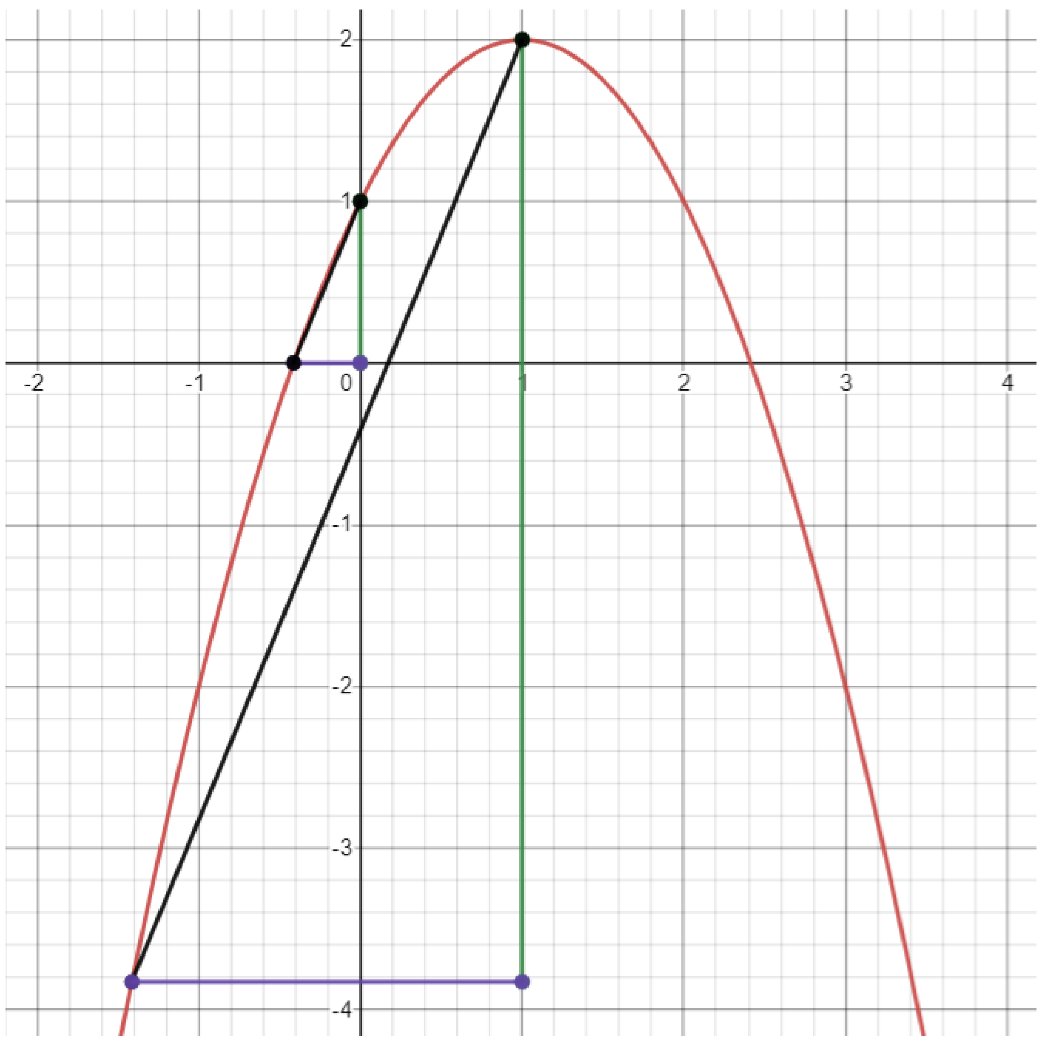
Image: Desmos

Define the slope at the point $(0, c)$.	$m_1 = b$
Define the slope at the point $(-k, 0)$.	$m_2 = \sqrt{b^2 - 4ac}$
Refer to the Secant-Slope Theorem.	$m_s = \dfrac{m_1 + m_2}{2}$
Apply the Secant-Slope Theorem to these two points.	$m_s = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
Refer to the definition of a slope and apply to c and k.	$\dfrac{\Delta y}{\Delta x} = \dfrac{c}{k}$
Equate with m_s.	$\dfrac{\Delta y}{\Delta x} = \dfrac{c}{k} = m_s = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
More simply.	$\dfrac{c}{k} = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
Solve for k.	$k = \dfrac{2c}{b + \sqrt{b^2 - 4ac}}$

Proof of K-Value by Similar Triangles

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

The value k can be solved using similar triangles. Consider a right triangle of the three points $(0,0)$, $(0, c)$, and $(-k, 0)$. A similar triangle can be constructed using the Tangent Point Y-Value Theorem, the Tangent Point X-Value Theorem, and the Secant-Vertex Theorem.



The large similar triangle is constructed using various theorems. Image: Desmos

The triangle exists with the vertexes $(0,0)$, $(-k,0)$ and $(0,c)$. To construct a similar triangle, the point corresponding to $(-k,0)$ must be found. Refer to Parabola Point Equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a}+c\right)$
Let the point in question be named Point T and defined as having the tangent with the slope m_T. Then the point equation for Point T is defined by	$\left(\frac{m_T-b}{2a}, \frac{{m_T}^2-b^2}{4a}+c\right)$
As described in the previous proof, the hypotenuse connecting $(-k,0)$ and $(0,c)$ has the slope defined by	$m_s = \frac{b+\sqrt{b^2-4ac}}{2}$
Since we have similar triangles, the slope from the vertex to Point T must have the same slope, such that	$m_v = m_s = \frac{b+\sqrt{b^2-4ac}}{2}$
Refer to the Secant-Vertex Theorem.	$m_v = \frac{m_2}{2}$
Since the slope at the vertex is 0, the slope of the secant is half that of the tangent of Point T.	$m_v = \frac{m_T}{2} = \frac{b+\sqrt{b^2-4ac}}{2}$
This gives the tangent slope of Point T.	$m_T = b+\sqrt{b^2-4ac}$
Using the Tangent Point X-Value Theorem and the Tangent Point Y-Value Theorem, the lengths of the legs of the similar triangle is obtained.	$l_x = -\frac{m_T}{2a}$

	$l_y = -\frac{{m_T}^2}{4a}$
The ratio of the similar triangles is given by	$\frac{k}{c} = \frac{l_x}{l_y}$
More exactly.	$\frac{k}{c} = \frac{-\frac{m_T}{2a}}{-\frac{{m_T}^2}{4a}}$
Cross multiply.	$\frac{k}{c} = -\frac{m_T}{2a} \cdot -\frac{4a}{{m_T}^2}$
Simplify.	$\frac{k}{c} = \frac{2}{m_T}$
Solve for k.	$k = \frac{2c}{m_T}$
Insert the value for m_T.	$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$

With the value of k ascertained, the 2C Quadratic Formula is confirmed.

$$x_0 = 0 - \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

$$x_0 = -\frac{b}{a} + \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

Parabolic Interval Equations

$$y(x+1) = y(x) + y'(x) + 1 \quad \{a = 1, n = 1\}$$

$$y(x+n) = y(x) + n(y'(x) + n) \quad \{a = 1, n \in \mathbb{R}\}$$

$$y(x+1) = y(x) + y'(x) + a \quad \{a \in \mathbb{R}, n = 1\}$$

$$y(x+n) = y(x) + n(y'(x) + an) \quad \{a \in \mathbb{R}, n \in \mathbb{R}\}$$

DEFINITIONS

a, b	The coefficients of the standard vertical parabola given by $y = ax^2 + bx + c$.
c	The constant of the standard vertical parabola given by $y = ax^2 + bx + c$.
n	The arbitrary interval from one point of a parabola to another.

Surmising the utility of these equations is not easy, since it will most often be more convenient to directly solve for $y(x)$ for any x. There is a teaching benefit for the teacher who would wish to demonstrate to students how the derivative of an equation changes the value of the equation. Additionally, these equations can be used to recover the constant c after integration.

Because the secondary equations are subsumed under the general equation, only the proof of the general equation is given here.

Reference the general equation.	$y(x+n) = y(x) + n(y'(x) + an)$
Express the left side of the equation.	$a(x+n)^2 + b(x+n) + c = y(x) + n(y'(x) + an)$
Express $y(x)$ and $y'(x)$.	$a(x+n)^2 + b(x+n) + c$ $= ax^2 + bx + c + n(2ax + b + an)$
Multiply n through.	$a(x+n)^2 + b(x+n) + c$ $= ax^2 + bx + c + 2axn + bn + an^2$
Group terms.	$a(x+n)^2 + b(x+n) + c$ $= ax^2 + 2axn + an^2 + bx + bn + c$
Factor a and b.	$a(x+n)^2 + b(x+n) + c$ $= a(x^2 + 2xn + n^2) + b(x+n) + c$
Factor the parenthetical.	$a(x+n)^2 + b(x+n) + c$ $= a(x+n)^2 + b(x+n) + c$

Recovering the Constant c

The constant c can be recovered using the parabolic interval equations with the two necessary givens being a point on the parabola and the derivative of the parabola. I will teach this by example.

Givens.	$P(4, 25)$ $y' = 4x + 6$
Refer to the parabolic interval equation with the interval given by $n = 1$.	$y(x+1) = y(x) + y'(x) + a$
Place the x values.	$y(5) = y(4) + y'(4) + a$
Express the derivative.	$y(5) = y(4) + 4(4) + 6 + a$
Integrate y'.	$\int y' = 2x^2 + 6x + c$
Place the integral, noting that $x = 5$.	$2(5)^2 + 6(5) + c$ $= y(4) + 4(4) + 6 + a$
Place the values for $y(4)$ and a.	$2(5)^2 + 6(5) + c$ $= 25 + 4(4) + 6 + 2$
Calculate.	$80 + c = 49$
Recover c.	$c = -31$
Derive the parent equation.	$y = 2x^2 + 6x - 31$

SECTION B

DELTA-COEFFICENT MATHEMATICS

Delta-Coefficient Mathematics

Lines have slopes defined by

$$m = \frac{(y_1 - y_2)}{(x_1 - x_2)}$$

The slope may be expressed as

$$m = \frac{\Delta y}{\Delta x}$$

Note that the interval is arbitrary. The slope is genuine as long as the interval for x and y is the same, and this is satisfied by using the same two points.

The slope can be expressed in Leibniz notation as

$$m = \frac{\mathrm{d}y}{\mathrm{d}x}$$

To distinguish the dy and dx values, the line

$$y = \frac{3}{5}x$$

may be rewritten as

$$5y = 3x$$

which gives the values

$$dx = 5$$

$$dy = 3$$

These values may have utility in certain operations.

In this discussion, I will distinguish delta values from derivatives.

A delta value:

$$dy = 3$$

A derivative:

$$\frac{dy}{dx} = \frac{3}{5}$$

I do not know the utility of Delta-Coefficient Mathematics. I would be glad to learn that it is useful to someone.

Origin Line Equations

$$dx \cdot y = dy \cdot x$$

Lines that pass through the origin are a convenient starting point. Consider the equation and the delta values:

$$5y = 3x$$

$$dx = 5$$

$$dy = 3$$

The delta values may be interpreted as follows: At every interval, an x-value change of 5 corresponds to a y-value change of 3. Thus, the following chart may be constructed:

$5y = 3x$	
$\boldsymbol{x}$ $\Delta x = 5$	$\boldsymbol{y}$ $\Delta y = 3$
0	0
5	3
10	6
15	9
20	12
$x = n \cdot \Delta x$	$y = n \cdot \Delta y$

Intercept Line Equations

$$dx \cdot y = dy \cdot x + 1$$

A line with intercepts outside the origin can be similarly arranged.

Given.	$2y - 8x = 4$
Divide by the constant.	$\frac{1}{2}y - 2x = 1$
Identify the intercepts.	$y_0 = \frac{1}{dx} = 2$ $x_0 = \frac{1}{dy} = -\frac{1}{2}$
Move the x term to the right side of the equation.	$\frac{1}{2}y = 2x + 1$
Identify your delta values.	$dx = \frac{1}{2}$ $dy = 2$

This line can be plotted in a similar fashion to a line passing through the origin, but special attention must be given the intercepts.

	$\frac{1}{2}y = 2x + 1$	
$\boldsymbol{n}$	$\boldsymbol{x}$ $\Delta x = \frac{1}{2}$	$\boldsymbol{y}$ $y_0 = 2$ $\Delta y = 2$
0	0	$0 + 2 = 2$
1	$\frac{1}{2}$	$2 + 2 = 4$
2	1	$4 + 2 = 6$
3	$\frac{3}{2}$	$6 + 2 = 8$
4	2	$8 + 2 = 10$
	$x = n \cdot \Delta x$	$y = n \cdot \Delta y + y_0$

Plane Equations

$$dydz \cdot x + dxdz \cdot y + dxdy \cdot z = 1$$

Find delta values.

Plane Equation	$Ax + By + Cz = D$
Example	$4x + 8y + 6z = 2$
Divide by the constant.	$dydz \cdot x + dxdz \cdot y + dxdy \cdot z = 1$
Although the delta values have changed, their ratios remain the same.	$2 \cdot x + 4 \cdot y + 3 \cdot z = 1$
Product delta values.	$dydz = 2$ $dxdz = 4$ $dxdy = 3$
Delta values can be solved without substitution. Multiply all coefficients.	$dx^2dy^2dz^2 = 2 \cdot 4 \cdot 3 = 24$
To solve for dx, first divide by $(dydz)^2$.	$dx^2 = \frac{24}{4} = 6$
Then take the square root. Do similarly for dy and dz.	$dx = 2.449$ $dy = 1.225$ $dz = 1.633$
The plane can now be expressed as	$1.225_y 1.633_z \cdot x + 2.449_x 1.633_z \cdot y + 2.449_x 1.225_y \cdot z = 1$

Finding Intercepts

DCE Plane Equation	$dydz \cdot x + dxdz \cdot y \ + dxdy \cdot z = 1$
Example	$2 \cdot x + 4 \cdot y \ + 3 \cdot z = 1$
Identify the intercepts.	$x_0 = \frac{1}{dydz}$ $y_0 = \frac{1}{dxdz}$ $z_0 = \frac{1}{dxdy}$
Example	$x_0 = \frac{1}{2}$ $y_0 = \frac{1}{4}$ $z_0 = \frac{1}{3}$

Finding Intercept Lines

DCE Plane Equation	$dydz \cdot x + dxdz \cdot y \ + dxdy \cdot z = 1$
Set $z = 0$.	$dydz \cdot x + dxdz \cdot y = 1$
Bring x term to the left side of the equation.	$dxdz \cdot y = -dydz \cdot x + 1$
To convert to y-intercept form, divide by $dxdz$.	$y = -\frac{dydz}{dxdz} \cdot x + \frac{1}{dxdz}$
Simplify.	$y = -\frac{dy}{dx} \cdot x + y_0$

Intercept Line Triangle

A simplified method for quickly plotting the intercept triangle on a plane, once dx, dy, and dz are known, is to apply the formula that follows.

DCE Plane Equation	$dydz \cdot x + dxdz \cdot y + dxdy \cdot z = 1$
Divide by $dxdydz$.	$\frac{x}{dx} + \frac{y}{dy} + \frac{z}{dz} = \frac{1}{dxdydz}$
Set variables equal to 0 to create the three lines.	$\frac{x}{dx} + \frac{y}{dy} = \frac{1}{dxdydz}$

Ellipsoid Equations

$$\frac{x^2}{dydz} + \frac{y^2}{dxdz} + \frac{z^2}{dxdy} = 1$$

$$\frac{x^2}{dy^2dz^2} + \frac{y^2}{dx^2dz^2} + \frac{z^2}{dx^2dy^2} = 1$$

Delta values for ellipsoids may be expressed in one of the forms shown above, though it is difficult, at least for me, to determine philosophically which is the more legitimate.

Form 1	$\frac{x^2}{dydz} + \frac{y^2}{dxdz} + \frac{z^2}{dxdy} = 1$
Example	$\frac{x^2}{10} + \frac{y^2}{20} + \frac{z^2}{40} = 1$
Product delta values.	$dydz = 10$ $dxdz = 20$ $dxdy = 40$
Find the delta values.	$dx = \sqrt{80} = 4\sqrt{5}$ $dy = \sqrt{20} = 2\sqrt{5}$ $dz = \sqrt{5}$
The ellipsoid can now be expressed as	$2\sqrt{5}_y\sqrt{5}_z \cdot x + 4\sqrt{5}_x\sqrt{5}_z \cdot y$ $+ 4\sqrt{5}_x 2\sqrt{5}_y \cdot z = 1$

The equation for the volume of an ellipsoid is

$$V = \frac{4}{3}\pi \cdot a \cdot b \cdot c$$

For the example above:

$$V = \frac{4}{3}\pi \cdot 10 \cdot 20 \cdot 40 = 33{,}510.32$$

The result of doubling dx:

$$2dx = 2\left(4\sqrt{5}\right) = 8\sqrt{5}$$

This results in the quadrupling of the volume.

$$V = \frac{4}{3}\pi \cdot 10 \cdot 40 \cdot 80 = 134{,}041.29$$

Doubling dx will consistently quadruple the volume.

$$V = \frac{4}{3}\pi \cdot (dydz) \cdot (dxdz) \cdot (dydx)$$

$$2dx : V = \frac{4}{3}\pi \cdot (dydz) \cdot (2dxdz) \cdot (dy2dx)$$

Form 2	$\frac{x^2}{dy^2dz^2}+\frac{y^2}{dx^2dz^2}+\frac{z^2}{dx^2dy^2}=1$
Example	$\frac{x^2}{10}+\frac{y^2}{20}+\frac{z^2}{40}=1$
Product delta values.	$dy^2dz^2=10$ $dx^2dz^2=20$ $dx^2dy^2=40$
Find the delta values.	$dx^2=\sqrt{80}\rightarrow dx=2.991$ $dy^2=\sqrt{20}\rightarrow dy=2.115$ $dz^2=\sqrt{5}\rightarrow dz=1.495$
The ellipsoid is expressed with squares.	$2.12_{y^2}1.50_{z^2}\cdot x+2.99_{x^2}1.50_{z^2}\cdot y$ $+2.99_{x^2}2.12_{y^2}\cdot z=1$

The result of doubling dx:

$$2dx=2(2.991)=5.981$$

This results in the volume increasing by 16 times.

$$V=\frac{4}{3}\pi\cdot 10\cdot 80\cdot 160=536{,}165.15$$

Doubling dx will consistently increase the volume by 16 times.

$$V = \frac{4}{3}\pi \cdot (dy^2 dz^2) \cdot (dx^2 dz^2) \cdot (dy^2 dx^2)$$

$$2dx : V = \frac{4}{3}\pi \cdot (dy^2 dz^2) \cdot (4dx^2 dz^2) \cdot (dy^2 4dx^2)$$

Parabola Equations

$$dx \cdot y = dy_1 \cdot x^2 + dy_2 \cdot x$$

A parabola may have multiple dy values. They operate independently. This is an important distinction. A dy value and dx value may operate in concert, but multiple dy values only interact in sums. Here n denotes the interval. For this example, a constant c will be neglected.

$3y = 7x^2 + 5x$					
n	$\boldsymbol{x}$ $\mathrm{d}x = 3$	$dy_1 = 7$	$dy_2 = 5$	$\boldsymbol{y}$	$(\boldsymbol{x}, \boldsymbol{y})$
0	0	0	0	0	$(0, 0)$
1	$(1 \cdot 3) = 3$	$(1 \cdot 3)(1 \cdot 7)$	$(1 \cdot 5)$	26	$(3, 26)$
2	$(2 \cdot 3) = 6$	$(2 \cdot 3)(2 \cdot 7)$	$(2 \cdot 5)$	94	$(6, 94)$
3	$(3 \cdot 3) = 9$	$(3 \cdot 3)(3 \cdot 7)$	$(3 \cdot 5)$	204	$(9, 204)$
4	$(4 \cdot 3) = 12$	$(4 \cdot 3)(4 \cdot 7)$	$(4 \cdot 5)$	356	$(12, 356)$
	$n \cdot dx$	$(n \cdot dx)(n \cdot dy_1)$	$n \cdot dy_2$	$n^2 dx dy_1$ $+ n dy_2$	

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