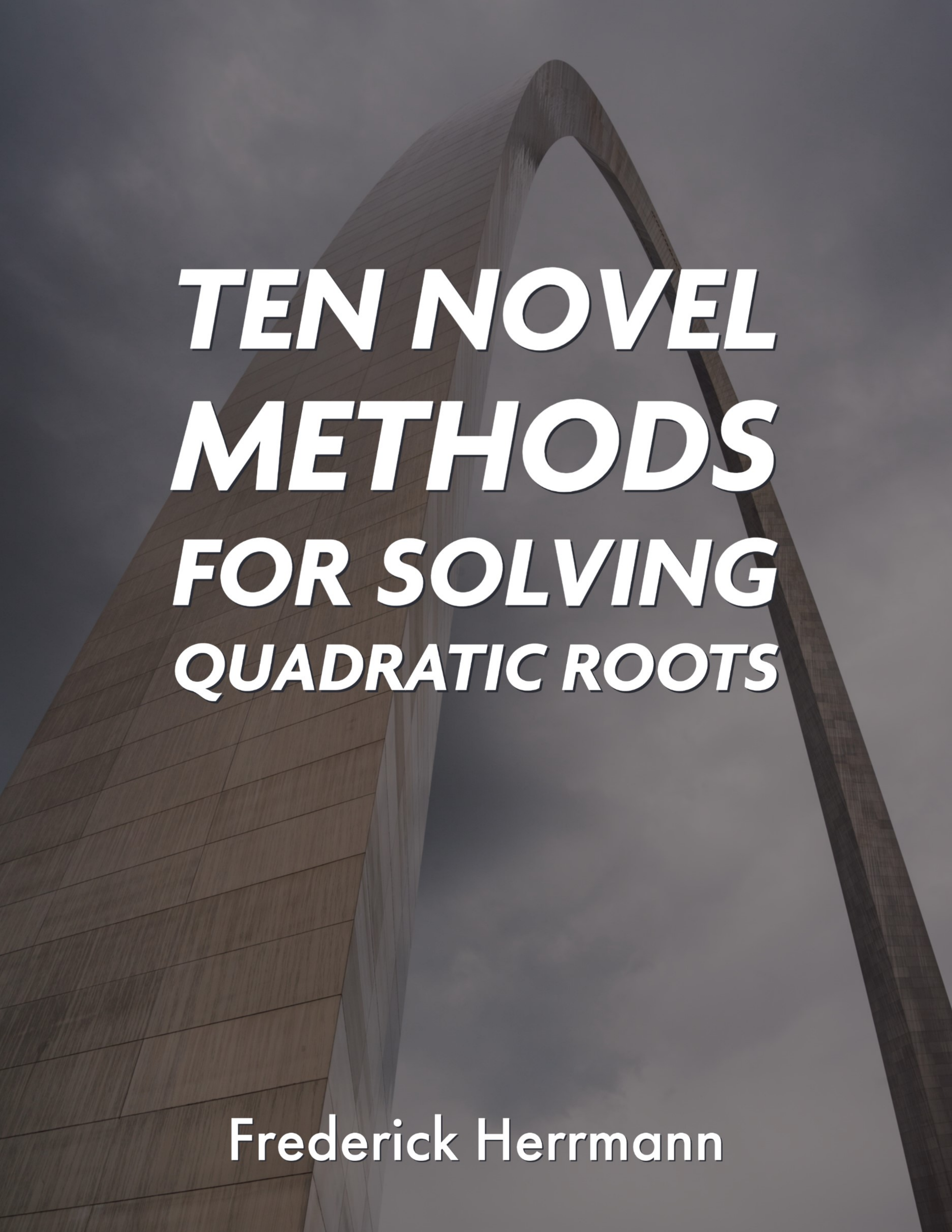


TEN NOVEL METHODS FOR SOLVING QUADRATIC ROOTS

Frederick Herrmann

TEN NOVEL METHODS FOR SOLVING QUADRATIC ROOTS

ʻE ko mākou Makua i loko o ka lani,
E hoʻāno ʻia kou inoa.

Frederick Herrmann
Makua Lani Christian Academy
Kailua-Kona, Hawaii
July 2023

Special thanks to Christian Williams
for his assistance with this work.

CONTENTS

INTRODUCTION

Ten Novel Methods for Solving Quadratic Roots is just what the title says it is: Ten methods for solving the roots of parabolas without using the standard form of the Quadratic Formula.

Several methods have practical value in solving quadratic roots as they are quicker or less error-prone than the Quadratic Formula.

Several methods are useful for teaching purposes, especially as they present two- and three-dimensional visualizations of what is really going on. Of course, we normally think of quadratics as parabolas—which they are—but in an equally valid sense, they are nothing more than the areas of a rectangles with defined perimeters! The three-dimensional methods bring in imaginary numbers, though the methods themselves are simple and straightforward.

Several methods are of mathematical interest only.

For my part, there are two big takeaways from this work: first, the relationship

$$|m| = L$$

meaning that the slope of a tangent line at a root of a parabola is equal in value to the distance between the parabola roots, and second,

$$\frac{L}{2} = \sqrt{y_m}$$

meaning (for the monic parabola) that half the distance between roots is equal to the square root of the "height" of the parabola. These mathematical theorems lead to several methods for solving quadratic roots.

A number of these methods were published in two of my previous papers, which the interested reader may access.

All these methods are "original to myself," and I have made an effort to see if anyone had trodden these steps before me. The chapter "Acknowledgements of Previous Studies" is included at the end of this book.

Mahalo!

DEFINITIONS

$y = ax^2 + bx + c$	The standard form of a vertical parabola.
x_m	The x-value of a parabola's line of symmetry; the x-value where the parabola has either its maximum or minimum y-value. This value is given by $x_m = \frac{-b}{2a}$.
y_m	The maximum or minimum y-value of the parabola. The y-value of the parabola vertex. Its value is given by $y(x_m)$.
$V(x_m, y_m)$	The vertex of the parabola.
(x, y, z)	The 3-space dimension, equivalent to the 2-space with imaginary $(x + zi, y)$. For the methods that utilize 3-space, the y-direction is always considered vertical.
x_0	The "roots" of an equation. The x-values of an equation where $y = 0$. The solutions to the quadratic formula.
L	The distance between roots.
Monic	The term that describes a parabola (often reduced) such that $a = 1$, taking the form: $y = x^2 + bx + c$.

#1

The Vertex Method

The Vertex Method is a simplified version of the Circular Paraboloid Method that ignores imaginary numbers. I tested this method with my Geometry classes (high school freshmen and sophomores) with a good deal of success. This method is valuable because it provides the students with the three numbers needed to graph a parabola, provided the roots are real. There are shortcuts for finding the y-value of the vertex that may be of interest to students; these are listed at the end of this book.

The Vertex Method is dependent on the relationship between the x-value distance between roots and the y_m value of the vertex. If we denote the roots of a parabola as $x_{0,1}$ and $x_{0,2}$ and the distance between the roots as $L = (x_{0,1} - x_{0,2})$, then in its simplest form for a monic parabola, the relationship exists:

$$\frac{L}{2} = \sqrt{y_m}$$

Therefore, the roots can be solved as

$$x_0 = x_m \pm \sqrt{y_m}$$

Developed April 2023.

The Vertex Method

$$x_0 = x_m \pm \sqrt{\left|\frac{y_m}{a}\right|}$$

Consider the parabola:

$$y = ax^2 + bx + c$$

Complete the following steps:

	$y = ax^2 + bx + c$
Find x_m.	$x_m = -\frac{b}{2a}$
Place the x_m value into the equation.	$y_m = a{x_m}^2 + bx_m + c$
Sum the right side.	$y_m = [numercial\ value]$
You now have the vertex.	(x_m, y_m)
Apply the x and y values of the vertex as shown. You may ignore all negative values.	$x_0 = x_m \pm \sqrt{\left\|\frac{y_m}{a}\right\|}$
	The problem is complete.

Example:

$$y = 2x^2 + 9x - 12$$

Complete the following steps:

	$y = 2x^2 + 9x - 12$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{9}{4} = -2.25$
Place the x_m value into the equation.	$y_m = 2(-2.25)^2 + 9(-2.25) - 12$
Sum the right side.	$y_m = -22.15$
You now have the vertex.	$(-2.25, -22.15)$
Apply the x and y values of the vertex. You may ignore all negative values under the radical.	$x_0 = -2.25 \pm \sqrt{\left\lvert\frac{-22.15}{2}\right\rvert}$
Solve the second term.	$x_0 = -2.25 \pm 3.33$
Solve.	$x_0 = 1.08, -5.58$
	The problem is complete.

Example with monic form and y_m shortcut.

$$x_0 = x_m \pm \sqrt{|-x_m^2 + c|}$$

Complete the following steps:

	$y = 2x^2 + 9x - 12$
Convert to monic form.	$y = x^2 + 4.5x - 6$
Find x_m.	$x_m = -\frac{b}{2} = -2.25$
Consider this y_m shortcut. Note that for a monic $a = 1$.	$y_m = -ax_m{}^2 + c$ $y_m = -x_m{}^2 + c \ (monic\ form)$
Use the y_m shortcut. You may ignore the negative in the square.	$y_m = -x_m{}^2 + c = -(2.25)^2 - 6$
Solve.	$y_m = -11.0625$
You now have the vertex of the monic parabola.	$(-2.25, -11.0625)$
Apply the x and y values of the vertex. You may ignore all negative values under the radical.	$x_0 = -2.25 \pm \sqrt{11.0625}$
	$x_0 = -2.25 \pm 3.33 = 1.08, -5.58$
	The problem is complete.

#2

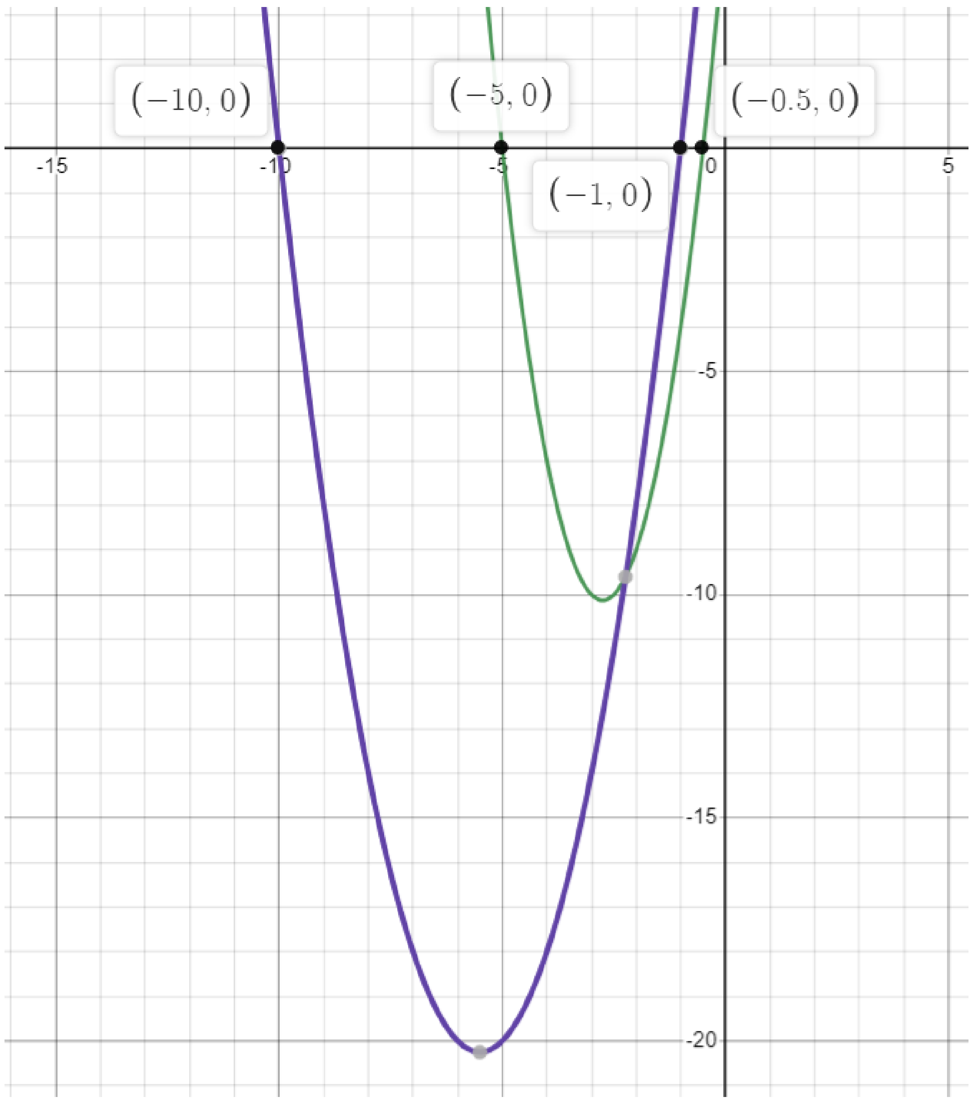
The Big Sister Method

The Big Sister Method is based on the concept that changing the a and c values by inverse factors does not change the value of the discriminant. The method allows a reduction to the monic form without creating fractions of b and c.

Developed June 2023.

The Big Sister Method

$$y = x^2 + bx + a \cdot c$$



The parabola $y = 2x^2 + 11x + 5$ (green) and its big sister $y = x^2 + 11x + 10$ (blue). The ratio of their solutions is a.

Image by the author using Desmos.

Consider the parabola:

$$y = ax^2 + bx + c_1$$

Complete the following steps:

	$y = ax^2 + bx + c_1$
Divide the first term by a and multiply the third term by a.	$y = x^2 + bx + a \cdot c = x^2 + bx + c_2$
Solve by factoring, or apply the reduced quadratic or the vertex method.	$x_0 = x_m \pm \sqrt{\lvert -x_m^2 + c_2 \rvert}$ The vertex method with the y_m shortcut.
You now have two solutions.	$x_{0,1}$ *and* $x_{0,2}$
Divide these solutions by a.	$\frac{x_{0,1}}{a}$ *and* $\frac{x_{0,2}}{a}$
	The problem is complete.

Example:

	$y = 3x^2 + 14x + 8$
Divide the first term by a and multiply the third term by a.	$y = x^2 + 14x + 24$
Solve by factoring.	$(x + 2)(x + 12) = 0$
You now have two solutions.	$-2\ and - 12$
Divide these solutions by a.	$-\frac{2}{3}\ and - 4$
	The problem is complete.

#3
The X-M Method

The X-M Method is the simple result of expressing the Quadratic Formula in terms of x-max or x-min (x_m), which is the x-value of the line of symmetry on which the vertex is located.

First published in my paper “Mathematical Exploratio I” (June 2021).

THE X-M METHOD

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

Consider the parabola

$$y = ax^2 + bx + c$$

Complete the following steps:

	$y = ax^2 + bx + c$
Find x_m.	$x_m = -\frac{b}{2a}$
Solve the discriminant.	$\Delta = {x_m}^2 - \frac{c}{a}$
Solve by adding to or subtracting from x_m.	$x_0 = x_m \pm \sqrt{\Delta}$

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

Example with Solution in 2-Space

	$y = -5x^2 + 7x + 2$
Find x_m.	$x_m = \frac{7}{10} = 0.7$
Using your calculator.	$0.7^2 + \frac{2}{5} = 0.89 \quad Ans^{0.5} = 0.943$
Solve by adding to or subtracting from x_m.	$x_0 = 0.7 \pm 0.943 = 1.643\ and - 0.243$

Example with Solution in 3-Space

	$y = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{5}{20} = -0.25$
Using your calculator.	$.25^2 - \frac{4}{10} = -0.3375 \quad Ans^{0.5} = 0.581i$
Solve by adding to or subtracting from x_m.	$x_0 = -0.25 \pm 0.581i$

PROOF OF X-MAX METHOD

$$x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$$

The equation above is arrived at through tinkering with the Quadratic Formula.

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

	$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Break the Quadratic Formula into two terms.	$x_0 = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
Square and root the denominator of the second term.	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{2^2 a^2}}$
Combine the second term under a single radical.	$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{2^2 a^2} - \frac{4ac}{2^2 a^2}}$
Simplify.	$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$
Substitute $x_m = -\frac{b}{2a}$.	$x = x_m \pm \sqrt{{x_m}^2 - \frac{c}{a}}$

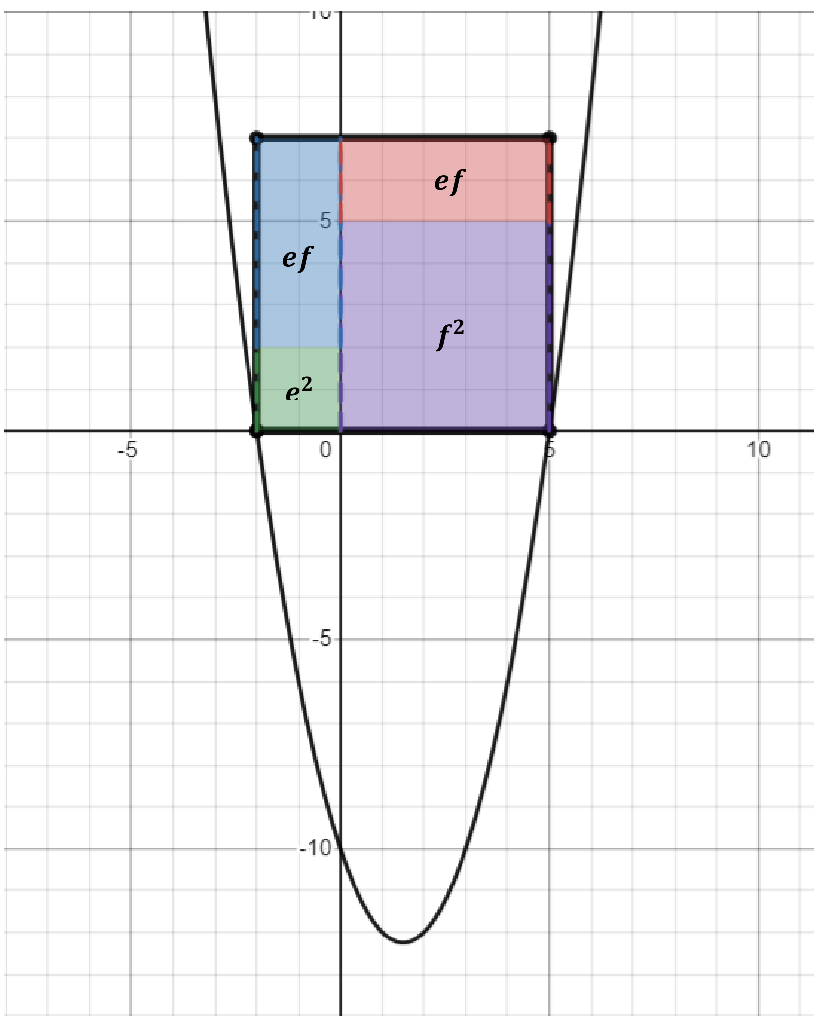
#4
The Black Box Method

The Black Box Method is essentially the geometric version the Quadratic Formula. The Black Box Method requires that b be taken as the difference (not the sum) of the roots. The method is valuable for instructional purposes. Due to the intricacies of this method, I begin with the proof and follow with an example. Please note that, though many instruction books include a geometric representation of "completing the square," this is not identical to the visual of the Black Box Method.

Developed June 2023.

The Black Box Method

$$x_0 = x_m \pm \frac{\sqrt{e^2 + 2ef + f^2}}{2}$$



The parabola $y = x^2 - 3x - 10$ has roots -2 and 5. The shaded green area represents 2^2, the shaded purple area represents 5^2, and the blue and red areas represent $2 \cdot 5$.

Image by the author using Desmos.

Consider the parabola:

$$y = ax^2 + bx + c$$

Proof:

	$y = ax^2 + bx + c$
Divide by a to convert to the monic form and allow for new values of b and c.	$y = x^2 + bx + c$
Consider b in terms of the roots e and f.	$b = e - f$
Consider c in terms of the roots e and f.	$c = ef$
Solve for L.	$L = \sqrt{b^2 + 4c}$
Begin by squaring b.	$b^2 = (e - f)^2 = e^2 - 2ef + f^2$
Multiply c by 4.	$4c = 4ef$
Add the terms. This is the total area of the box.	$b^2 + 4c = e^2 + 2ef + f^2$
Take the square root to find the length of a side of the box. This is the distance between roots.	$L = \sqrt{e^2 + 2ef + f^2}$
Obtain x_m.	$x_m = -\frac{b}{2}$
Your solutions are:	$x_0 = x_m \pm \frac{L}{2}$

Example:

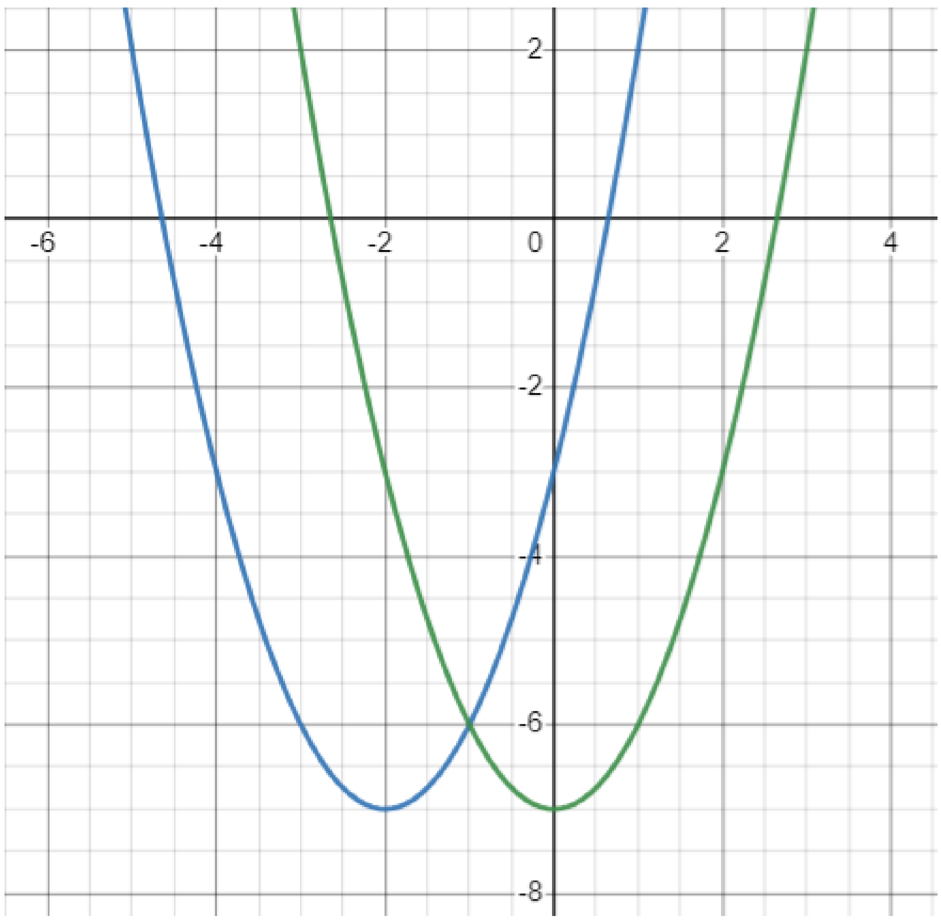
	$y = x^2 - 3x - 10$
Begin by squaring b.	$b^2 = 9$
Multiply c by 4. The area $2ef$ is considered positive.	$4c = \|-40\| = 40$
Add the terms. This is the area of the box.	$b^2 + 4c = 49$
Take the square root to find the length of a side of the box.	$L = \sqrt{49} = 7$
Obtain x_m.	$x_m = -\frac{b}{2} = 1.5$
Your solutions are:	$x_0 = x_m \pm \frac{L}{2} = 1.5 \pm 3.5 = -2, 5$

#5
The Twin Method

The Twin Method creates a new parabola with the same length L (distance between roots) as the original. The new parabola is easily solved because it has the feature $b = 0$. The discriminants of both parabolas have the same value.

Developed June 2023.

$$c_2 = -\frac{b^2}{4} + c_1$$



The Twin Method creates a new parabola (green) centered on the origin that has the same length (distance between roots) as the original (blue). Solving the new parabola provides the second term for the roots of the first parabola.

Image created by author using Desmos.

Consider the parabola

$$y = ax^2 + bx + c$$

Complete the following steps:

	$y = ax^2 + bx + c_1$
Convert to monic form and allow for new values of b and c_1.	$y = x^2 + bx + c_1$
Use this process to solve for c_2.	$c_2 = -\frac{b^2}{4} + c_1$
Your twin parabola is:	$y = x^2 + c_2$
Find x_m for your original parabola.	$x_m = -\frac{b}{2}$
Your solutions are:	$x_0 = x_m \pm \sqrt{\lvert c_2 \rvert}$

Example:

	$y = x^2 + 4x - 3$
Use this process to solve for c_2.	$c_2 = -\frac{16}{4} - 3 = -7$
Your twin parabola is:	$y = x^2 - 7$
Find x_m for your original parabola.	$x_m = -\frac{4}{2} = -2$
Your solutions are:	$x_0 = -2 \pm \sqrt{\lvert -7 \rvert} = 0.551, 5.449$

#6
The Circular Paraboloid Method

The Circular Paraboloid Method makes use of circular paraboloids. The method is slightly easier than using hyperbolic paraboloids, though it likewise relies on imaginary numbers.

First published in my paper “The 3-Space Method of Solving Parabolic Roots” (September 2021).

The Circular Paraboloid Method

$$-az^2 = ax^2 + bx + c$$

Consider the parabola:

$$y = ax^2 + bx + c$$

Complete the following steps:

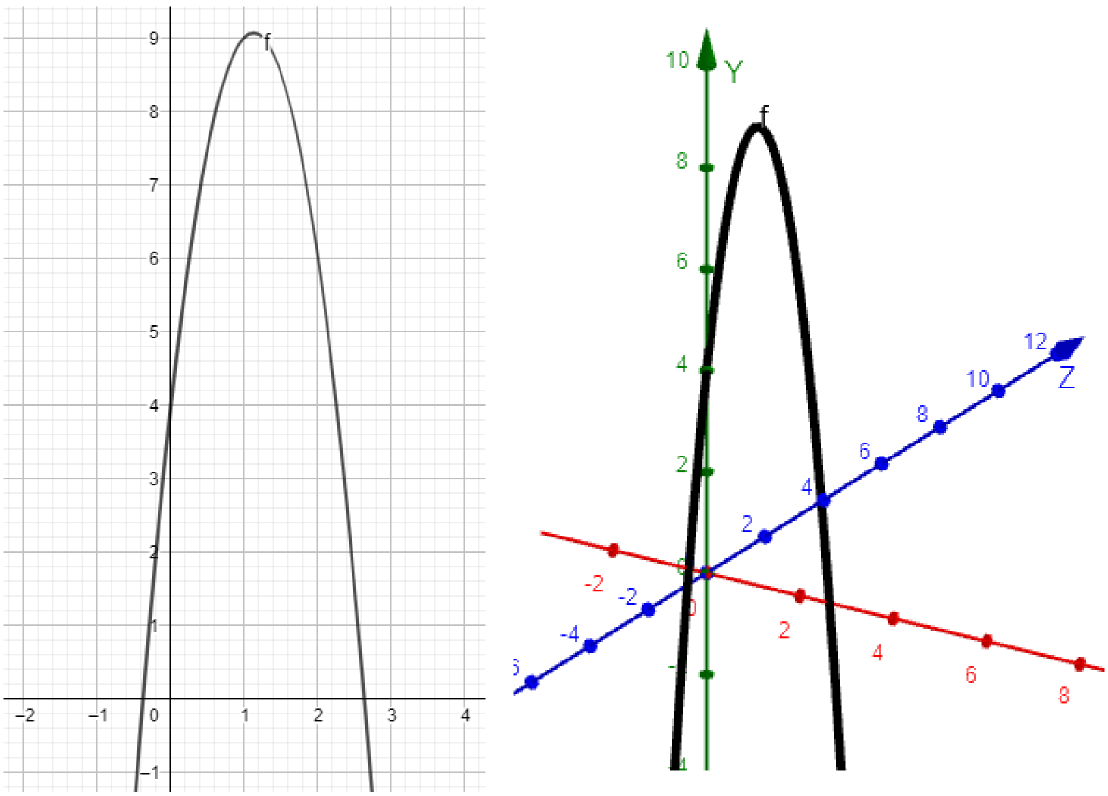
	$y = ax^2 + bx + c$
Add the z-term to create the paraboloid.	$y = ax^2 + bx + c + az^2$
Set $y = 0$.	$0 = ax^2 + bx + c + az^2$
Move the z-term to the left side of the equation.	$-az^2 = ax^2 + bx + c$
Find x_m.	$x_m = -\dfrac{b}{2a}$
Place the x_m value into the equation.	$az^2 = a{x_m}^2 + bx_m + c$
Sum the right side.	$az^2 = y_m$
You now have the vertex.	(x_m, y_m)
Solve for z.	$z = \sqrt{\dfrac{y_m}{a}}$
The solutions are:	$x_0 = x_m \pm z$

Example with Real Solutions

The parabola.	$y = -4x^2 + 9x + 4$
Add $-4z^2$.	$y = -4x^2 + 9x + 4 - 4z^2$
Let $y = 0$.	$0 = -4x^2 + 9x + 4 - 4z^2$
Move $4z^2$ to the left side of the equation.	$4z^2 = -4x^2 + 9x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = \frac{9}{8} = 1.125$
Place x_m into the equation and solve.	$4z^2 = -4(1.125)^2 + 9(1.125) + 4 = 9.0625$
You now have the vertex.	$V(1.125, 9.0625)$
Solve for z.	$z = \sqrt{\frac{9.0625}{4}} = \pm 1.505$
The solutions are:	$x_0 = 1.125 \pm 1.505 = 2.63\ and - 0.38$

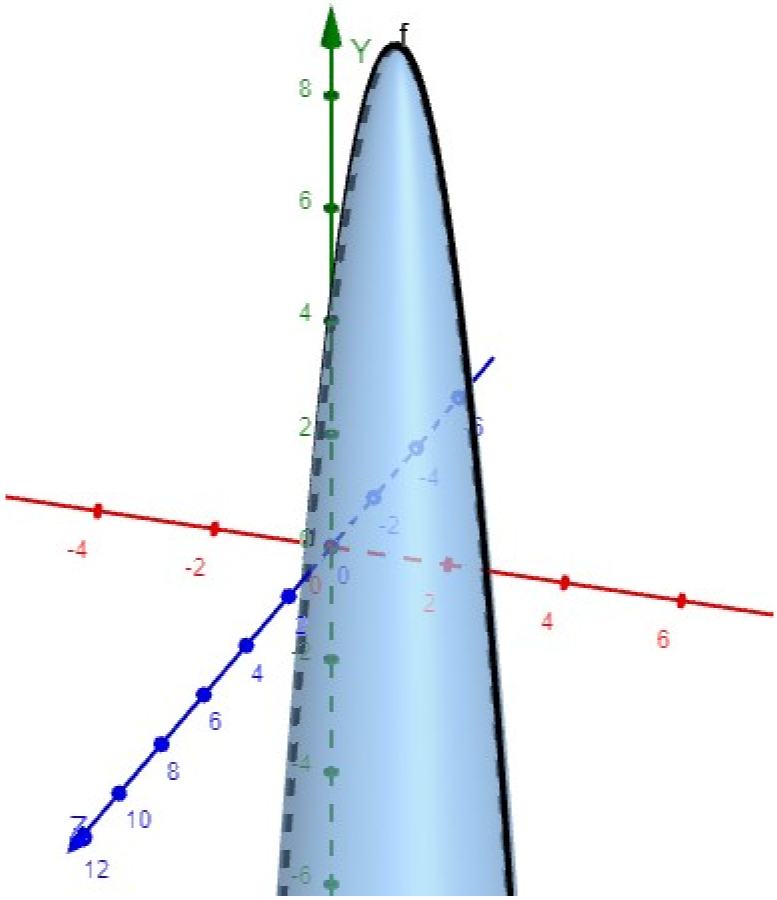
Example with Real Solutions: Images

The parabola.	$y = -4x^2 + 9x + 4$



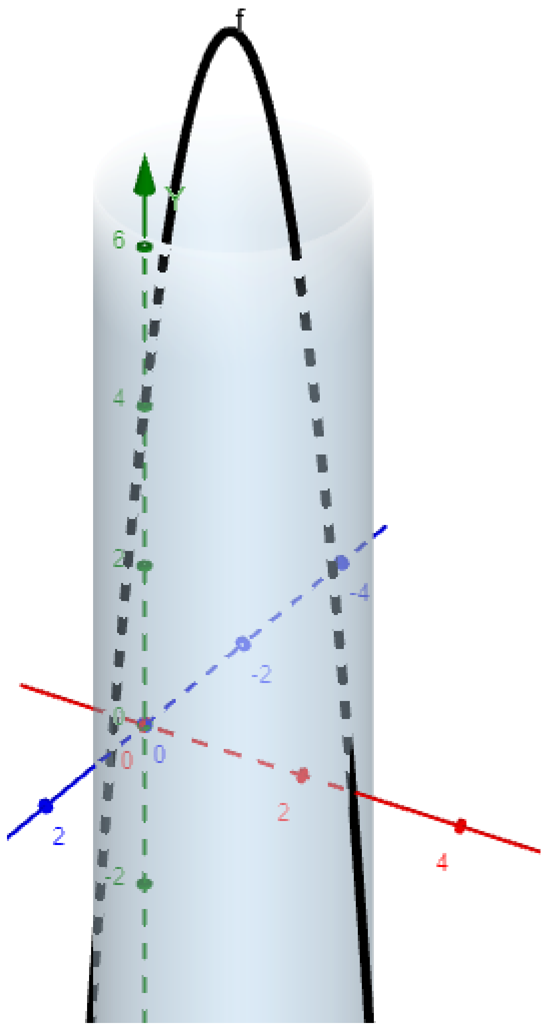
Created by author with GeoGebra. geogebra.org

Add $-4z^2$.	$y = -4x^2 + 9x + 4 - 4z^2$



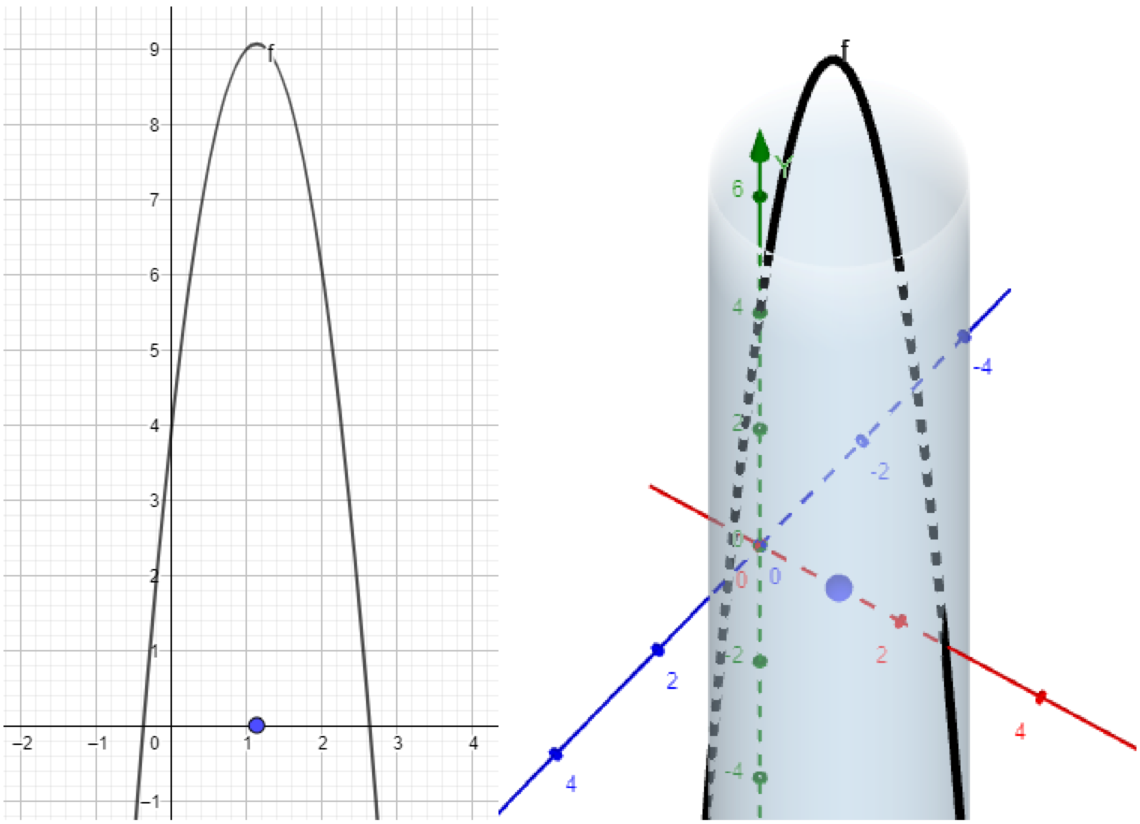
Created by author with GeoGebra. geogebra.org

Let $y = 0$.	$0 = -4x^2 + 9x + 4 - 4z^2$
Move $4z^2$ to the left side of the equation.	$4z^2 = -4x^2 + 9x + 4$



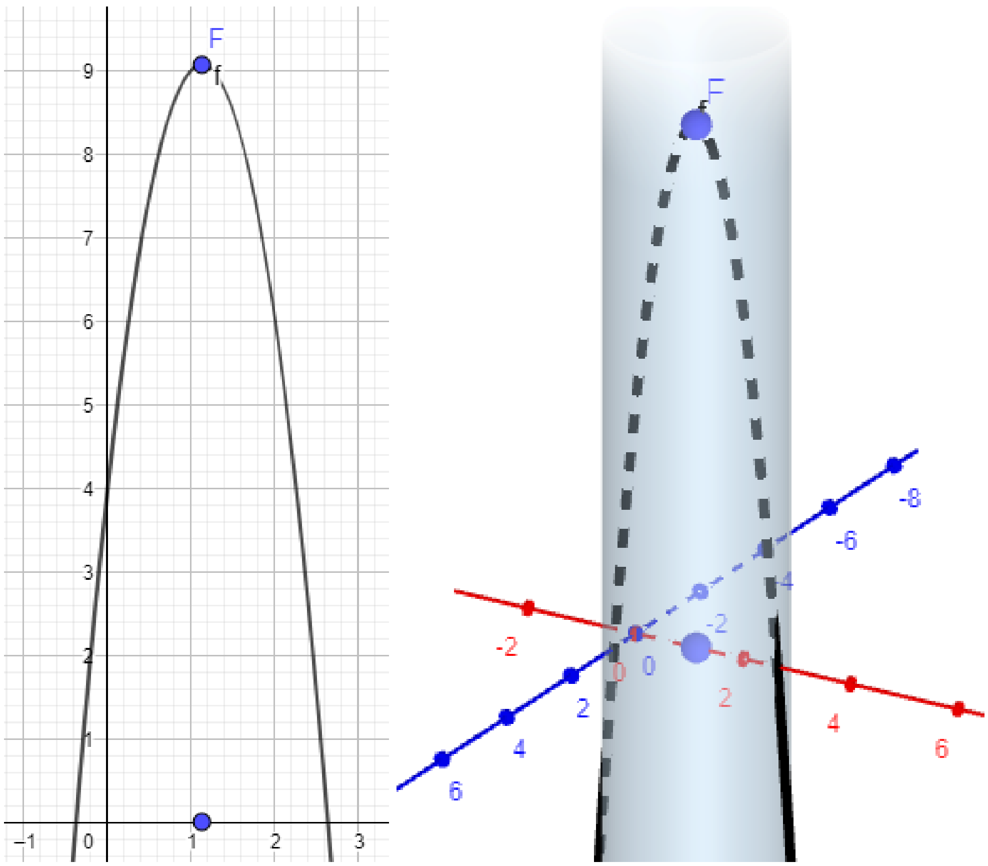
Created by author with GeoGebra. geogebra.org

Find x_m.	$x_m = -\frac{b}{2a} = \frac{9}{8} = 1.125$



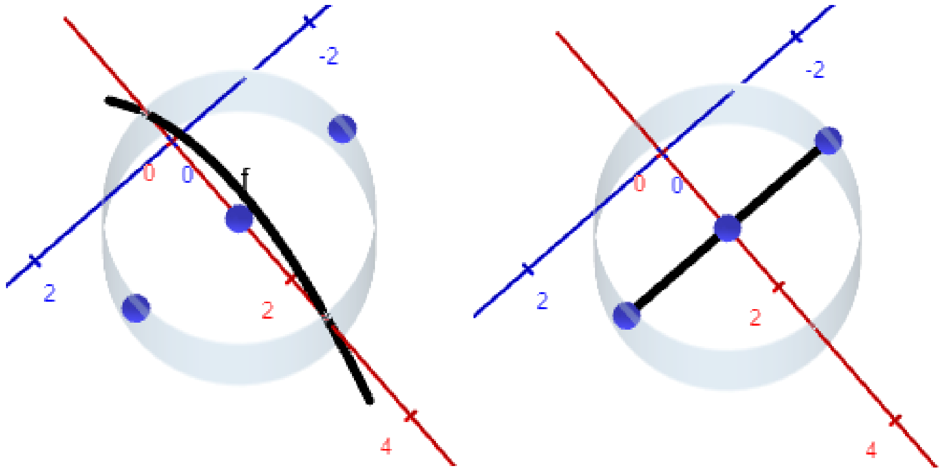
Created by author with GeoGebra. geogebra.org

Place x_m into the equation and solve.	$4z^2 = -4(1.125)^2 + 9(1.125) + 4 = 9.0625$
You now have the vertex.	$V(1.125, 9.0625)$



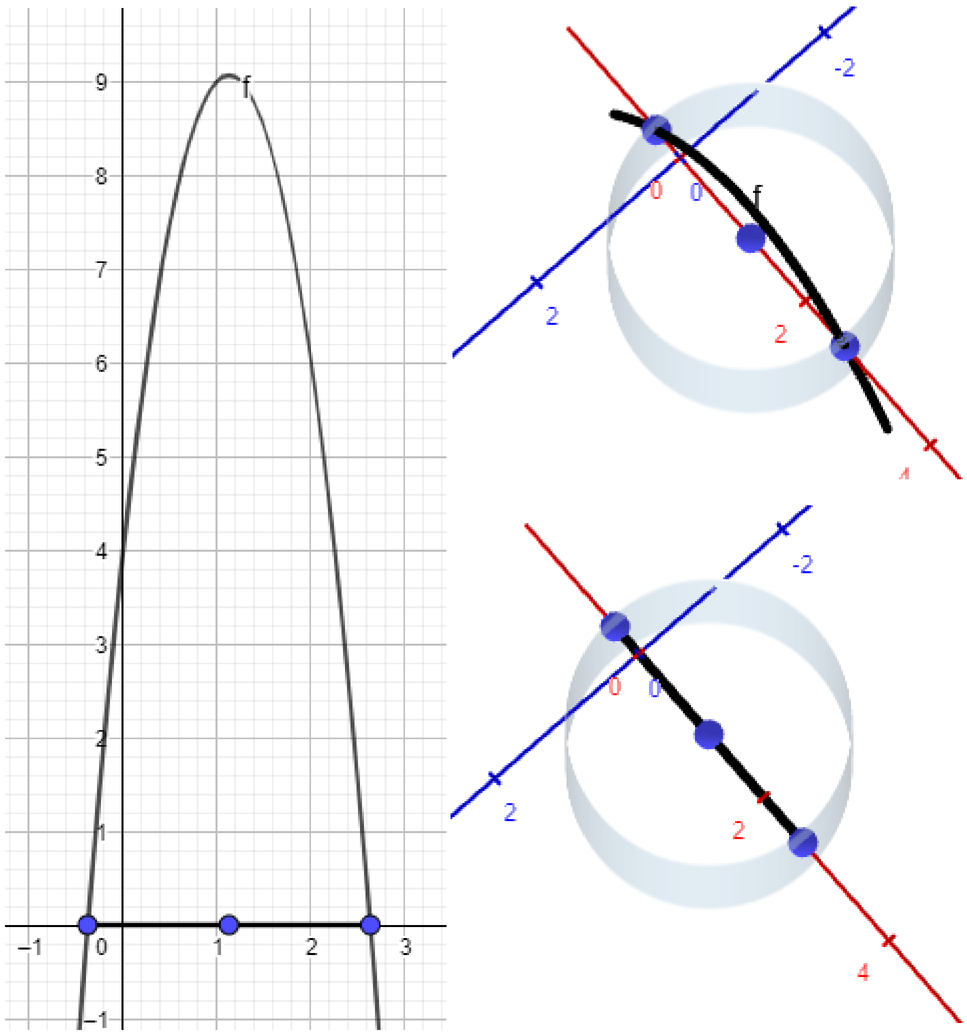
Created by author with GeoGebra. geogebra.org

Solve for z.	$z = \sqrt{\frac{9.0625}{4}} = \pm 1.505$



Created by author with GeoGebra. geogebra.org

The solutions are:	$x_0 = 1.125 \pm 1.505 = 2.63\ and - 0.38$



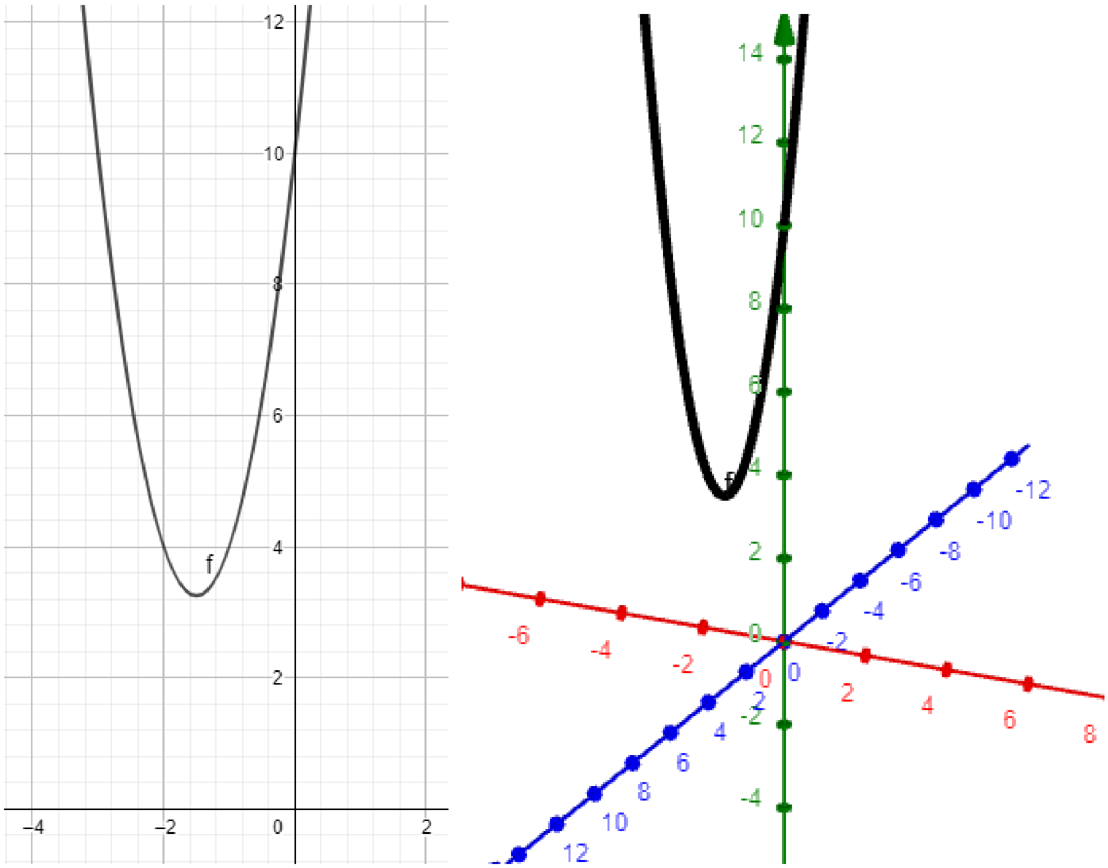
Created by author with GeoGebra. geogebra.org

Example with Imaginary Solutions

The parabola.	$y = 3x^2 + 9x + 10$
Add $3z^2$.	$y = 3x^2 + 9x + 10 + 3z^2$
Let $y = 0$.	$0 = 3x^2 + 9x + 10 + 3z^2$
Move $3z^2$ to the left side of the equation.	$-3z^2 = 3x^2 + 9x + 10$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{9}{6} = -1.5$
Place x_m into the equation and solve.	$-3z^2 = 3(-1.5)^2 + 9(-1.5) + 10 = 3.25$
You now have the vertex.	$V(-1.5, 3.25)$
Solve for z.	$z = \sqrt{\frac{3.25}{-3}} = \pm 1.041i$
The solutions are:	$x_0 = -1.5 \pm 1.041i$

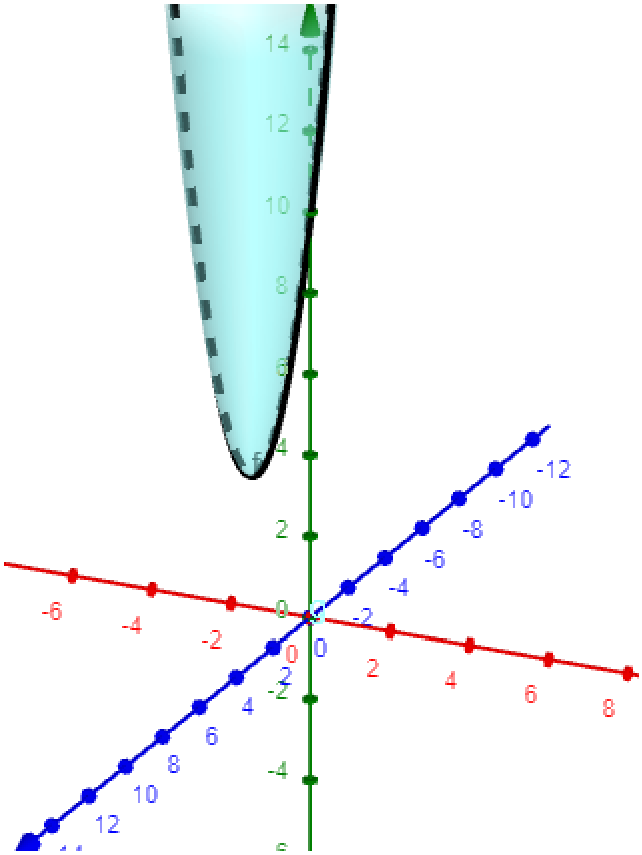
Example with Imaginary Solutions: Images

The parabola.	$y = 3x^2 + 9x + 10$



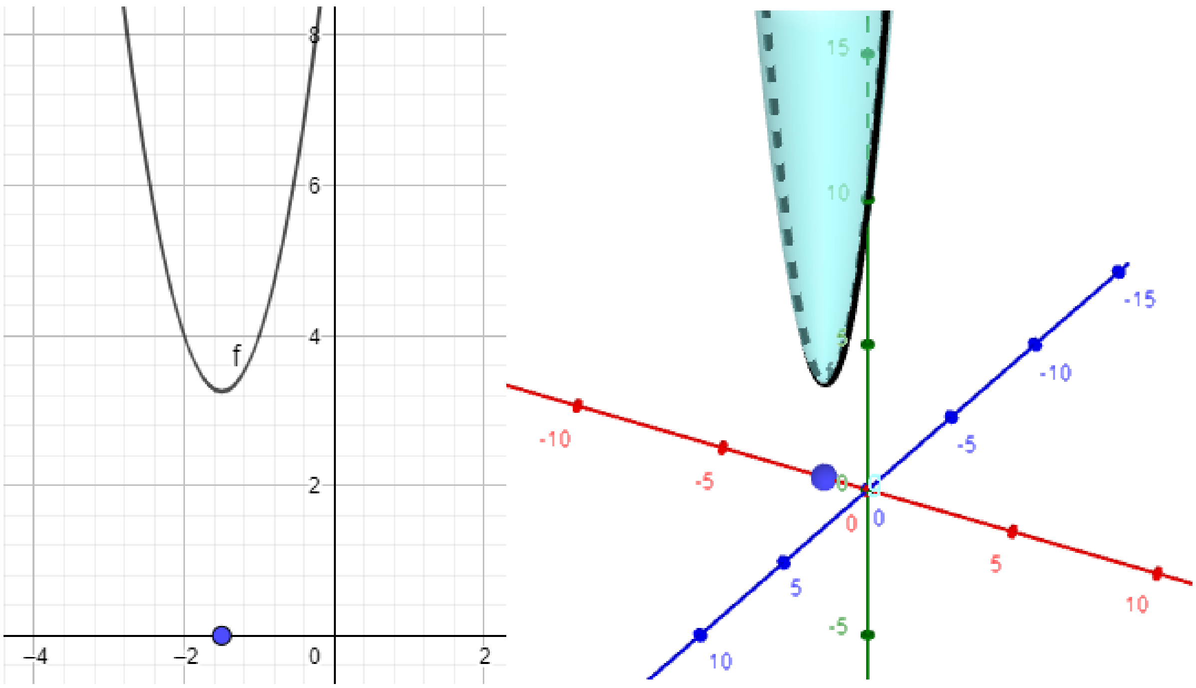
Created by author with GeoGebra. geogebra.org

Add $3z^2$.	$y = 3x^2 + 9x + 10 + 3z^2$



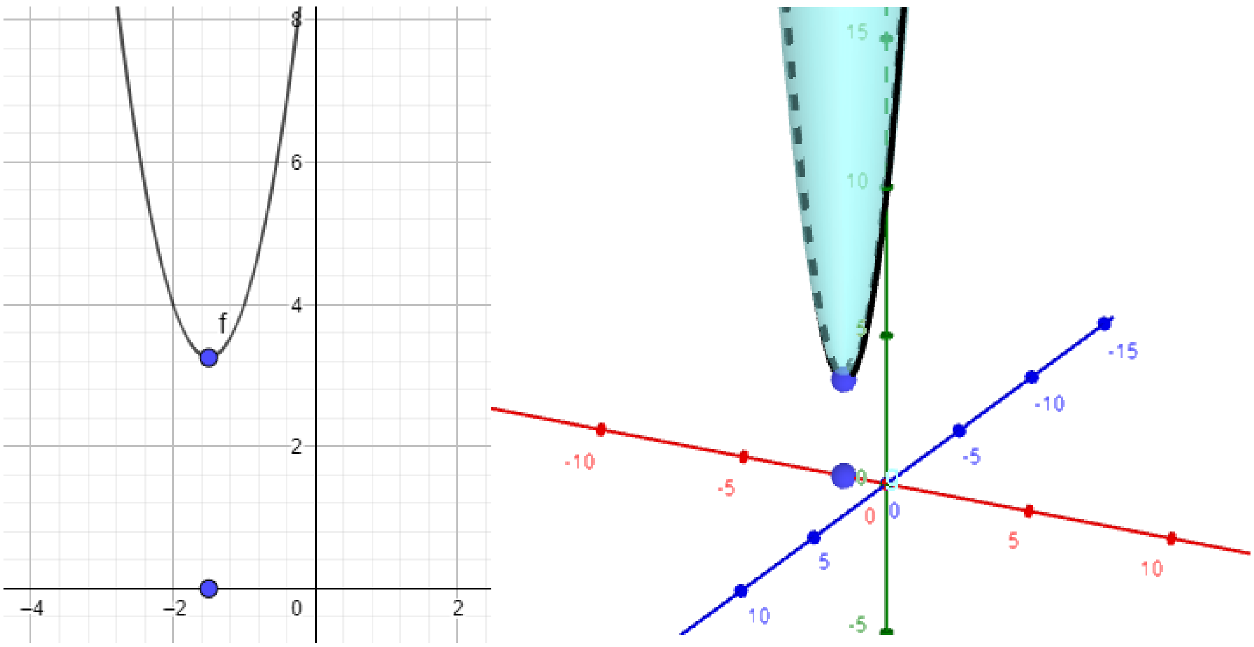
Created by author with GeoGebra. geogebra.org

Let $y = 0$.	$0 = 3x^2 + 9x + 10 + 3z^2$
Move $3z^2$ to the left side of the equation.	$-3z^2 = 3x^2 + 9x + 10$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{9}{6} = -1.5$



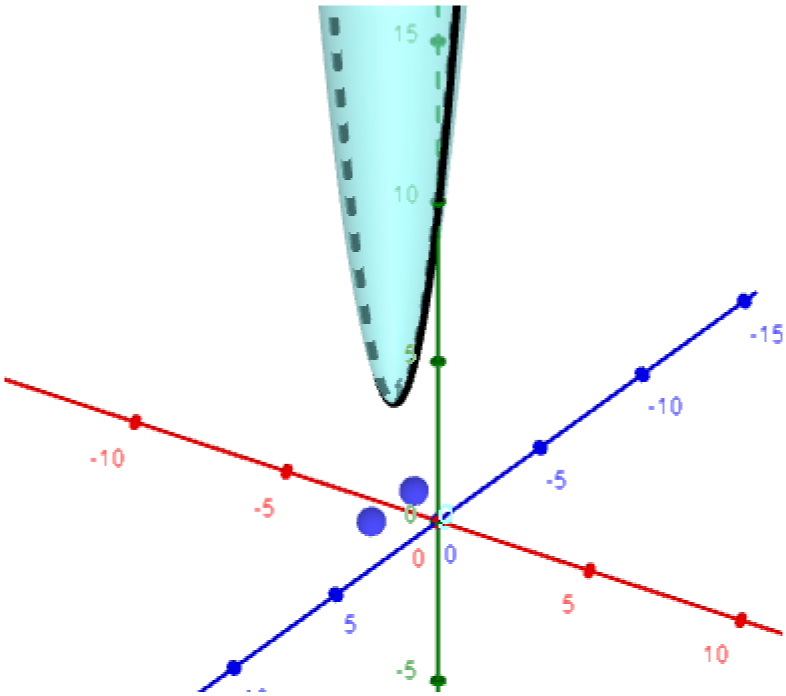
Created by author with GeoGebra. geogebra.org

Place x_m into the equation and solve.	$-3z^2 = 3(-1.5)^2 + 9(-1.5) + 10 = 3.25$
You now have the vertex.	$V(-1.5, 3.25)$



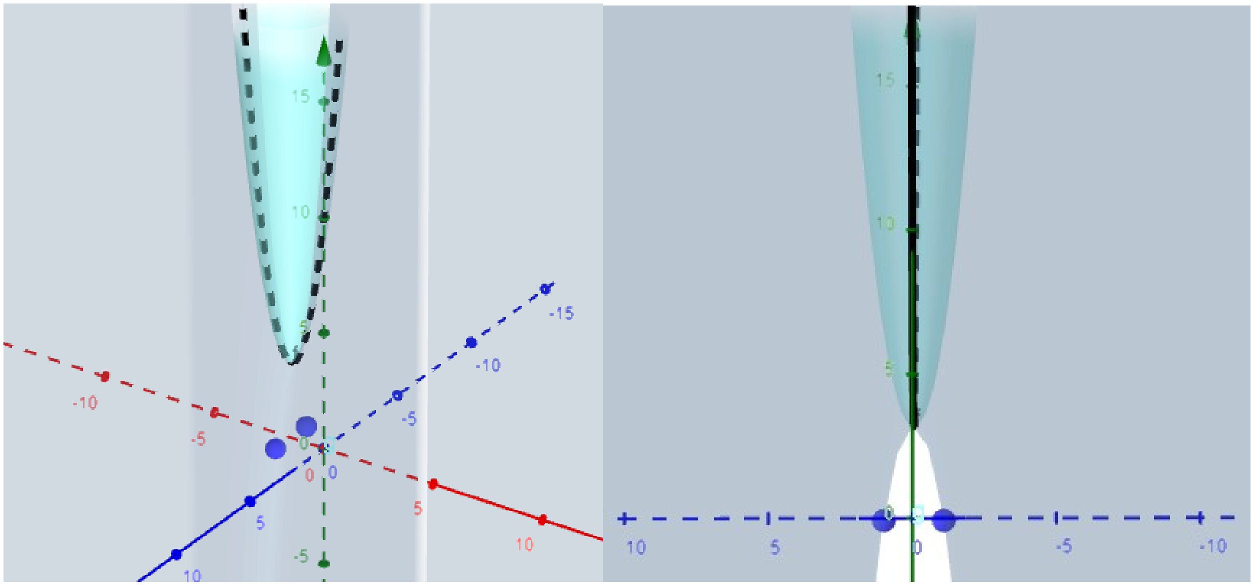
Created by author with GeoGebra. geogebra.org

Solve for z.	$z = \sqrt{\frac{3.25}{-3}} = \pm 1.041i$
The solutions are:	$x_0 = -1.5 \pm 1.041i$



Created by author with GeoGebra. geogebra.org

Solutions Images with Hyperbolic Paraboloid in Gray



Created by author with GeoGebra. geogebra.org

#7
Hyperbolic Paraboloid Method

The Hyperbolic Paraboloid Method makes use of imaginary numbers in three dimensional space. The hyperbolic paraboloid is the most pure 3-space representation of the parabola in the author's opinion because it flows naturally from the mathematics of parabolas.

First published in my paper "Mathematica Exploratio I" (June 2021).

THE HYPERBOLIC PARABOLOID METHOD

$$az^2 = ax^2 + bx + c$$

Consider the parabola

$$y = ax^2 + bx + c$$

Complete the following steps:

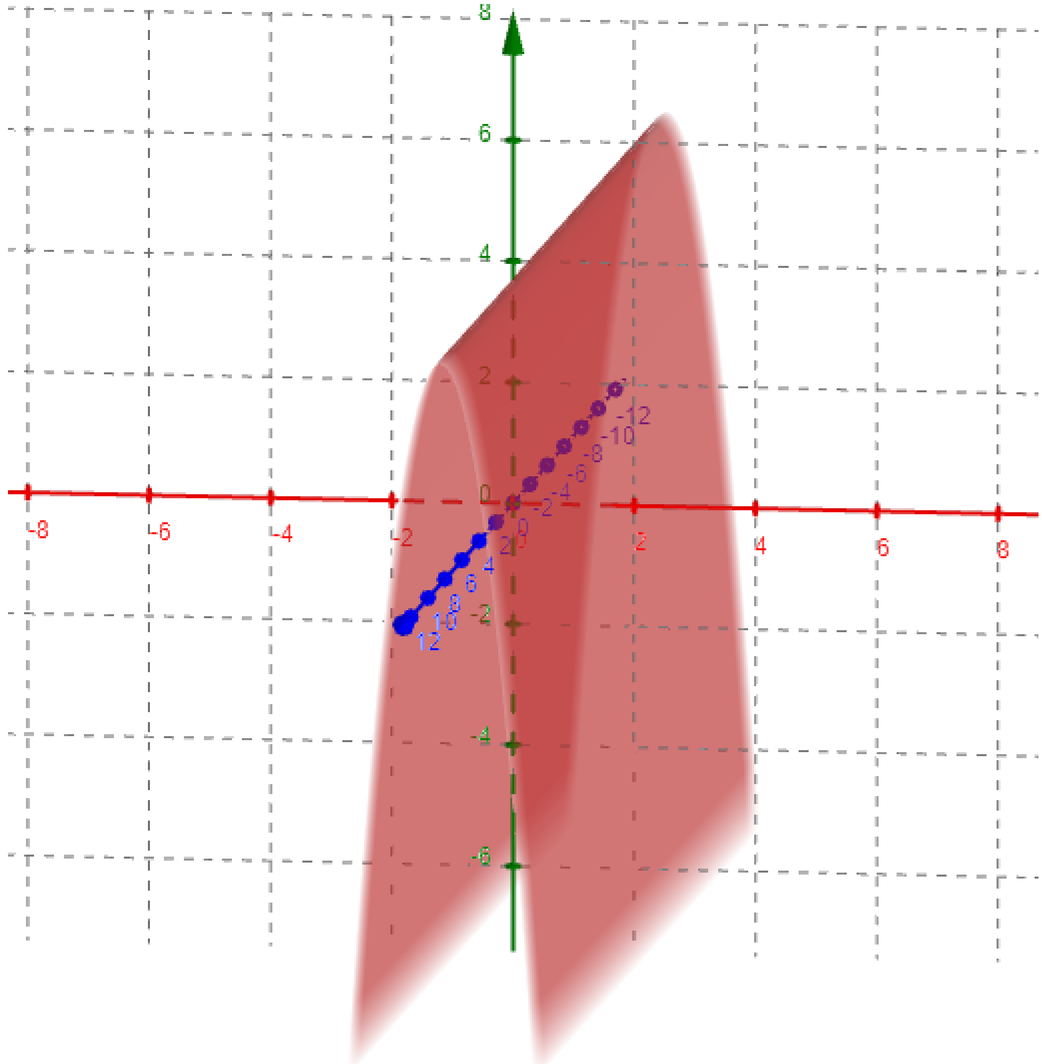
	$y = ax^2 + bx + c$
Replace y with az^2.	$az^2 = ax^2 + bx + c$
Find x_m.	$x_m = -\frac{b}{2a}$
Place the x_m into the equation.	$az^2 = ax_m{}^2 + bx_m + c = y_m$
You now have the vertex.	(x_m, y_m)
Solve for z.	$z = \sqrt{\frac{y_m}{a}}$
If z is imaginary, the solutions are:	$x_0 = x_m \pm z$
If z is real, the solutions are:	$x_0 = x_m \pm zi$ or $(x_m, 0, \pm z)$

Example with Solution in 2-Space

	$y = -5x^2 + 7x + 2$
Replace y with az^2.	$-5z^2 = -5x^2 + 7x + 2$
Find x_m.	$x_m = -\frac{b}{2a} = \frac{7}{10} = 0.7$
Place x_m into the equation and solve.	$-5z^2 = -5(0.7)^2 + 7(0.7) + 2 = 4.45$
You now have the vertex.	$V(0.7, 4.45)$
Solve for z.	$z = \sqrt{\frac{4.45}{-5}} = 0.943i$
If z is imaginary, the solutions are:	$x_0 = 0.7 \pm 0.943 = 1.643\ and - 0.243$
If z is real, the solutions are:	z is not real

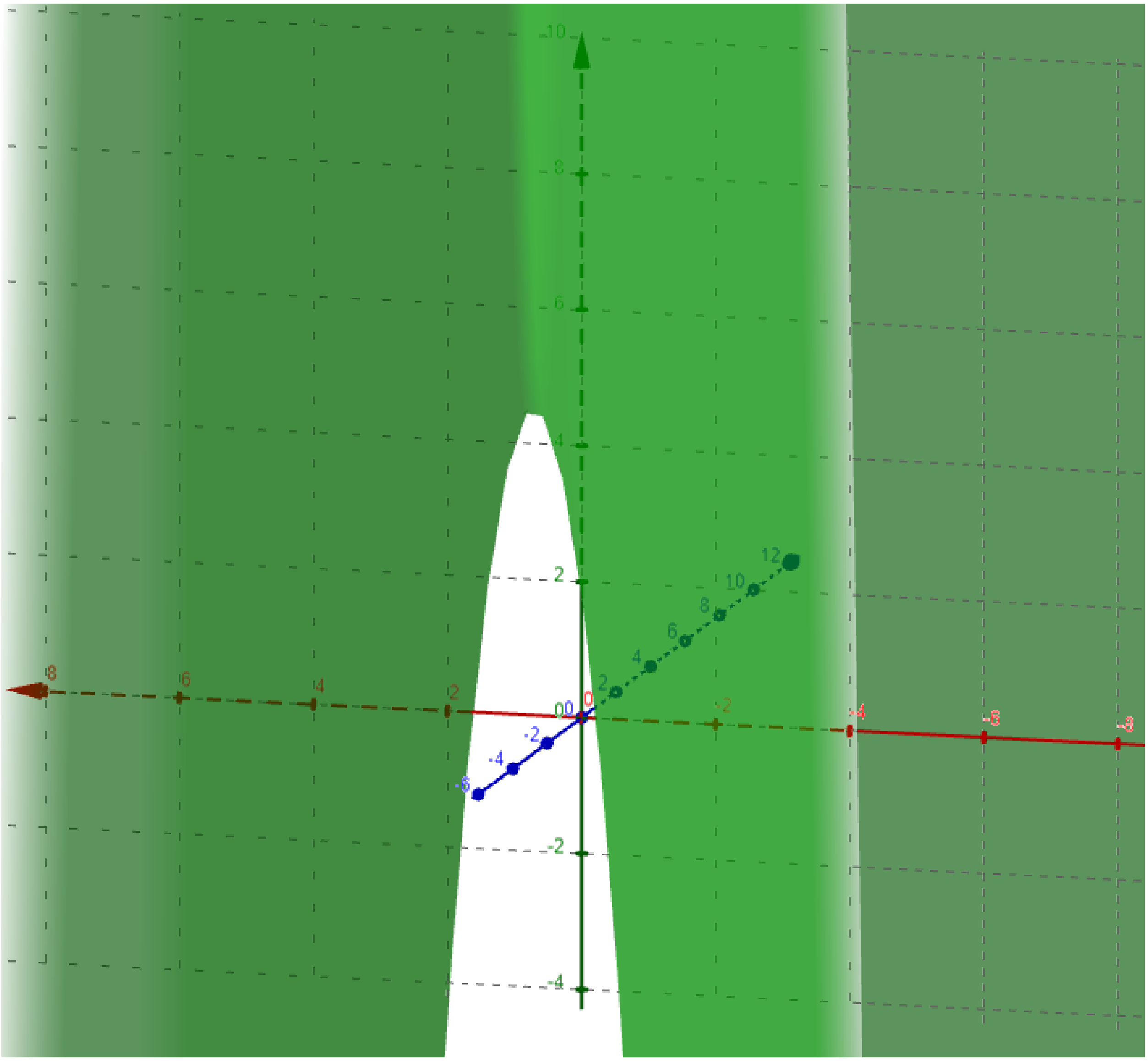
Example with Solution in 2-Space: Images

	$y = -5x^2 + 7x + 2$



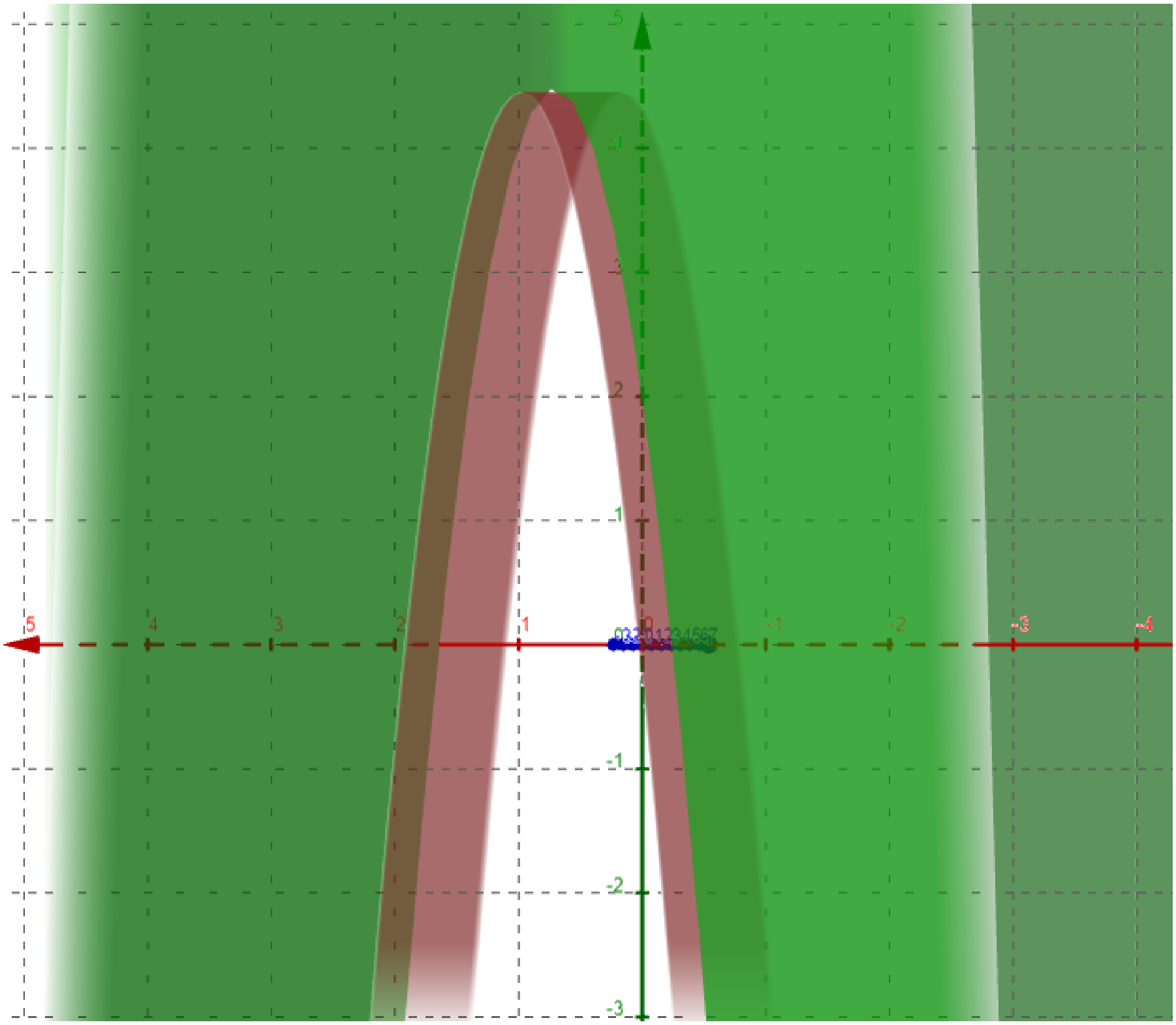
Created by author with GeoGebra. geogebra.org

Add $-az^2$ to the equation.	$y = -5x^2 + 7x + 2 + 5z^2$



Created by author with GeoGebra. geogebra.org

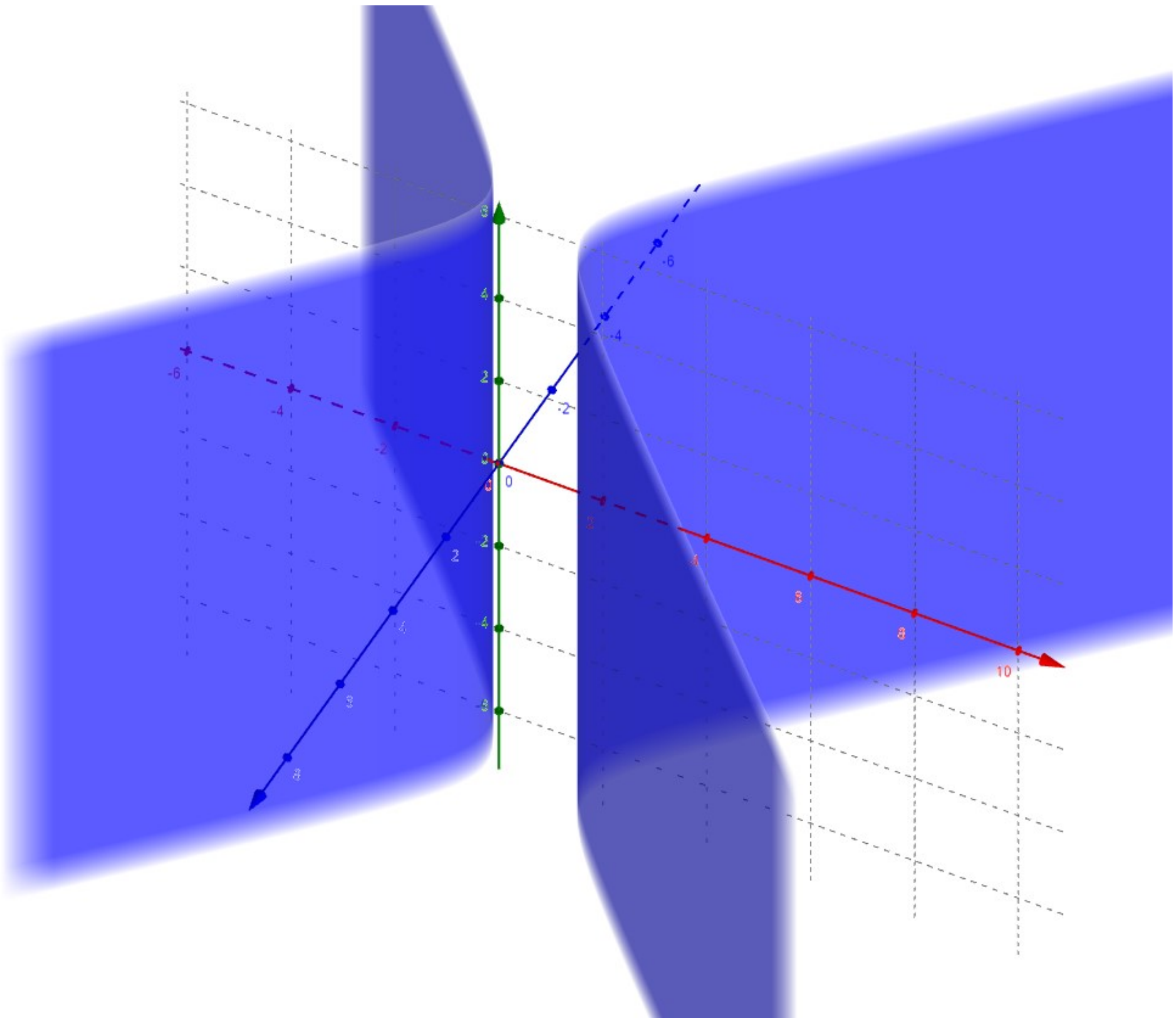
The hyperbolic paraboloid appears when $y \neq 0$, so this step should not be skipped if the teacher is demonstrating this visual. The hyperbolic paraboloid has the saddle shape and is symmetric about its axes.



Created by author with GeoGebra. geogebra.org

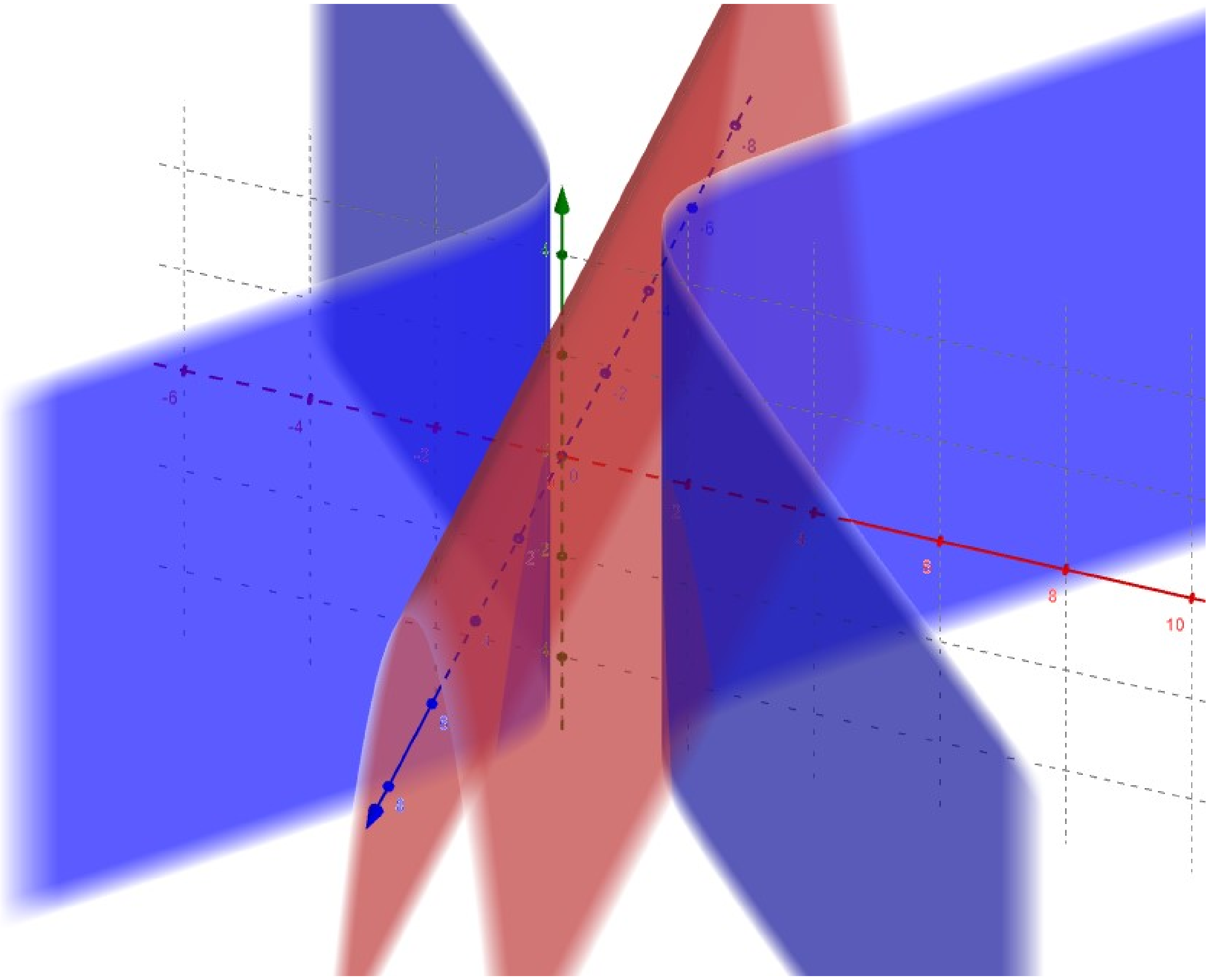
Notice that the parabola fits snuggly within the hyperbolic paraboloid.

Let $y = 0$ and shift az^2 to the left side of the equation.	$-5z^2 = -5x^2 + 7x + 2$



Created by author with GeoGebra. geogebra.org

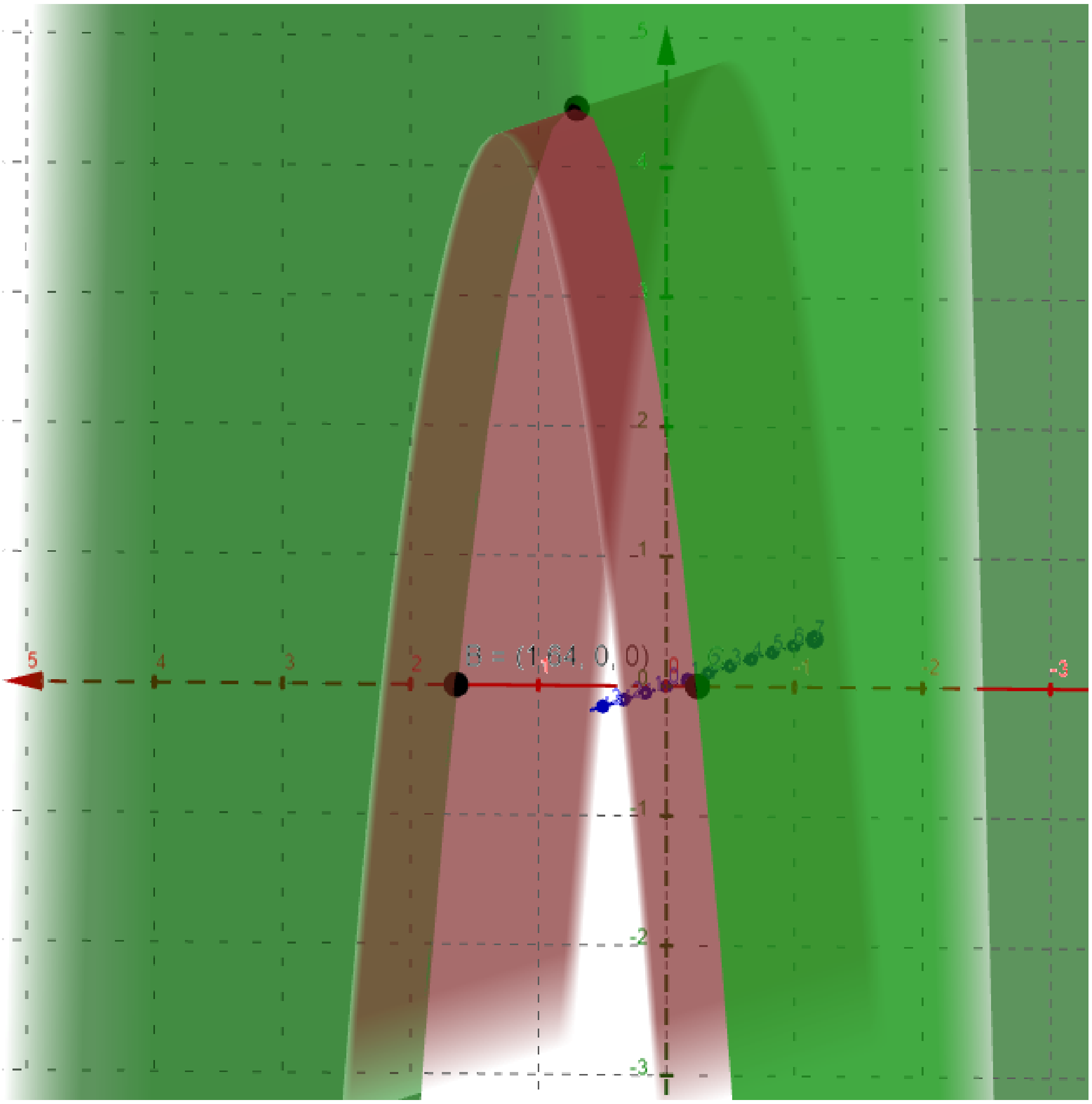
A hyperbolic cylinder is formed.



Created by author with GeoGebra. geogebra.org

The hyperbolic cylinder shares the roots of the parabola on the x-axis.

Solve for x_m and y_m to find the vertex.	$V(0.7, 4.45)$
Solve for z. If z is imaginary, the solutions are:	$x_0 = 0.7 \pm 0.943 = 1.643\ and - 0.243$



Created by author with GeoGebra. geogebra.org

Solving for the vertex and roots shows their positions on the hyperbolic paraboloid.

Example with Solution in 3-Space

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Place x_m into the equation and solve.	$az^2 = 10(-0.25)^2 + 5(-0.25) + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

3-Space Method with y_m Shortcuts

$$y_m = -a{x_m}^2 + c$$

$$y_m = \frac{b}{2}x_m + c$$

Applying the first y_m shortcut to the 3-Space Method

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Use first y_m shortcut.	$az^2 = -10(-0.25)^2 + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

Applying the second y_m shortcut to the 3-Space Method

	$y = 10x^2 + 5x + 4$
Replace y with az^2.	$az^2 = 10x^2 + 5x + 4$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{5}{20} = -0.25$
Use second y_m shortcut.	$az^2 = \frac{5}{2}(-0.25) + 4 = 3.375$
You now have the vertex.	$V(-0.25, 3.375)$
Solve for z.	$z = \sqrt{\frac{3.375}{10}} = 0.581$
If z is imaginary, the solutions are:	z is not imaginary
If z is real, the solutions are:	$x_0 = -0.25 \pm 0.581i$ or $(-0.25, 0, \pm 0.581)$

PROOF OF HYPERBOLIC PARABOLOID METHOD

$$y = ax^2 + bx + c - az^2$$

The equation above is constructed as a 3-space equivalent of a parabola given by:

$$y = ax^2 + bx + c$$

Finding roots for a parabola occurs by definition when $y = 0$. The proof proceeds with the following steps:

	$y = ax^2 + bx + c$
Replace 2-Space equation with 3-Space equation.	$y = ax^2 + bx + c - az^2$
Set $y = 0$.	$0 = ax^2 + bx + c - az^2$
Move az^2 to the left side of the equation.	$az^2 = ax^2 + bx + c$
Solve for z.	The z-roots are found by definition at $y = 0$ and $x = x_m$.

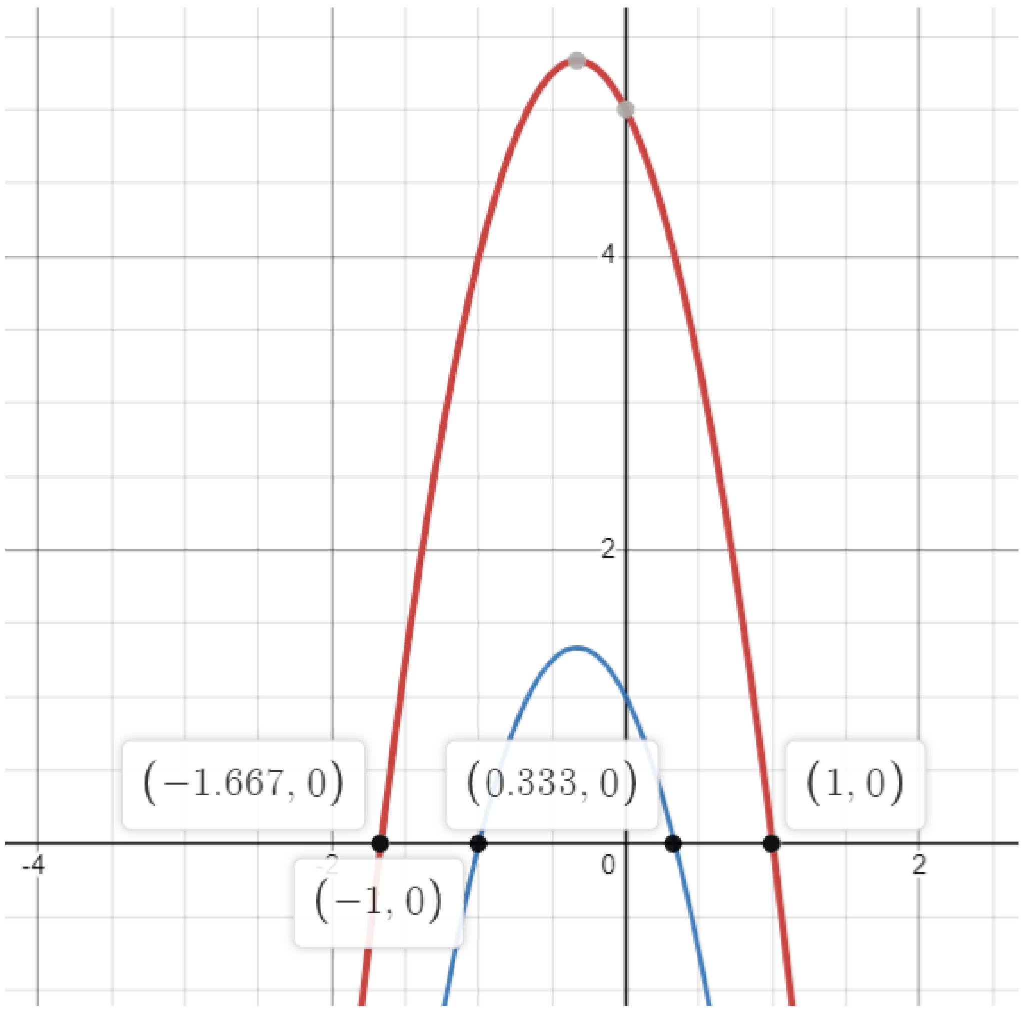
#8
The Companion Method

The Companion Method is based on a unique relationship of the coefficients a, b, and c. It generates a new parabola with a height ratio that can be used to solve the roots of the original parabola. This method uses a number of special equations that are unique to the companion parabola.

Developed June 2023.

The Companion Method

$$x_0 = x_m \pm \frac{R \cdot L_2}{2}$$



The Companion Method utilizes a height-to-length ratio. Companion parabolas have the roots -1 and $-\frac{c}{a}$ and the height $y_m = \frac{(c-a)^2}{-4a}$. This image displays the parabolas $y_1 = -3x^2 - 2x + 5$ (red) and its companion $y_2 = -3x^2 - 2x + 1$ (blue).

Image by author created with Desmos.

Consider the parabola:

$$y_1 = ax^2 + bx + c$$

Complete the following steps:

	$y_1 = ax^2 + bx + c$
Create a companion parabola by changing the value of c such that $b = a + c$.	$y_2 = ax^2 + bx + (c + k)$ $y_2 = ax^2 + bx + c_2$
The solutions of the roots of y_2 are:	$x_{0,1} = -1, x_{0,2} = -\frac{c_2}{a}$
The distance L_2 between the two roots is:	$L_2 = 1 + \frac{c_2}{a}$
Find the height (y_m) of y_2.	$y_m = \frac{(c_2 - a)^2}{-4a}$
Determine the square root of the ratio of the heights of the two parabolas.	$R = \sqrt{\frac{y_m + k}{y_m}} = \sqrt{1 + \frac{k}{y_m}}$
The distance L_1 between the roots of y_1 is given by:	$L_1 = R \cdot L_2$
Find x_m.	$x_m = -\frac{b}{2a}$
The solutions are:	$x_0 = x_m \pm \frac{L_1}{2}$

Example:

$$y_1 = -3x^2 - 2x + 5$$

Complete the following steps:

	$y_1 = -3x^2 - 2x + 5$
Create a companion parabola by changing the value of c such that $b = a + c$.	$y_2 = -3x^2 - 2x + (5 - 4)$ $y_2 = -3x^2 - 2x + 1$
The solutions of the roots of y_2 are:	$x_{0,1} = -1, x_{0,2} = \frac{1}{3}$
The distance L_2 between the two roots is:	$L_2 = 1 + \frac{c_2}{a} = \frac{4}{3} = 1.333$
Find the height (y_m) of y_2.	$y_m = \frac{(c_2 - a)^2}{-4a} = \frac{16}{12} = \frac{4}{3} = 1.333$
Determine the square root of the ratio of the heights. Use $k = 4$.	$R = \sqrt{1 + \frac{k}{y_m}} = \sqrt{1 + \frac{4}{1.333}} = 2$
The distance L_1 between the roots of y_1 is given by:	$L_1 = 2 \cdot L_2 = 2 \cdot 1.333 = 2.666$
The half-length is:	$\frac{L_1}{2} = \frac{2.666}{2} = 1.333$
Find x_m.	$x_m = -\frac{b}{2a} = -\frac{1}{3} = -0.333$
The solutions are:	$x_0 = x_m \pm \frac{L_1}{2} = -0.333 \pm 1.333 = -1.666, 1$

#9
The 2C Method

The 2C Method, unlike all the others, does not makes use of the x-value of the line of symmetry (x_m). Instead, it starts at the values $x = 0$ and $x = -\frac{b}{a}$ and works outwards from them. There are two proofs for the 2c Method. Both proofs employ theorems that I developed and published in "Mathematica Exploratio I."

First published in my paper "Mathematical Exploratio I" (June 2021).

The 2C Quadratic Formula

$$x_0 = 0 - \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

$$x_0 = -\frac{b}{a} + \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

This alternative to the Quadratic Formula may lack utility for the student as it is every bit as complex; however, one never knows when or for whom something may have utility.

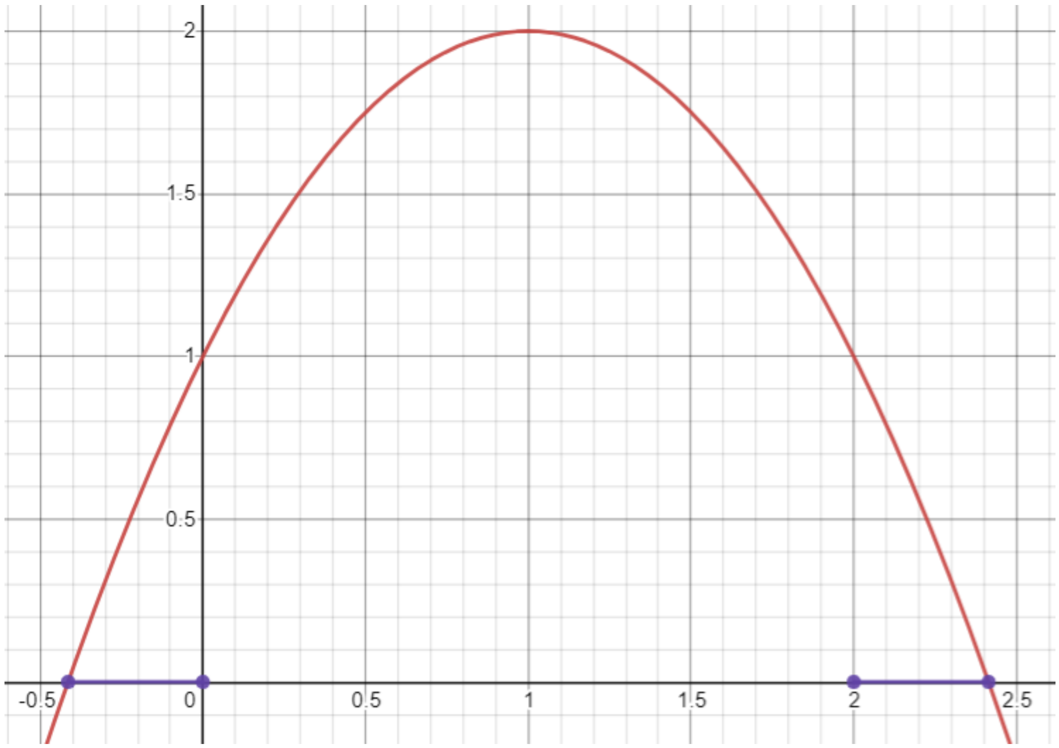
For convenience, the second term of the 2C Formula will be defined as k, giving

$$x_0 = 0 - k$$

$$x_0 = -\frac{b}{a} + k$$

Thus, to prove the 2C Formula, all that is necessary is to show that

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$



The distance k shown in purple.

Image created by author using Desmos.

Proof of K-Value by Slopes

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

The value k can be derived using the Secant-Slope Theorem and the ratio of $\frac{c}{k}$. The Secant-Slope Theorem will provide the slope of the secant (black), which must be equal to the ratio of c (green) over k (blue). The only unknown is k.

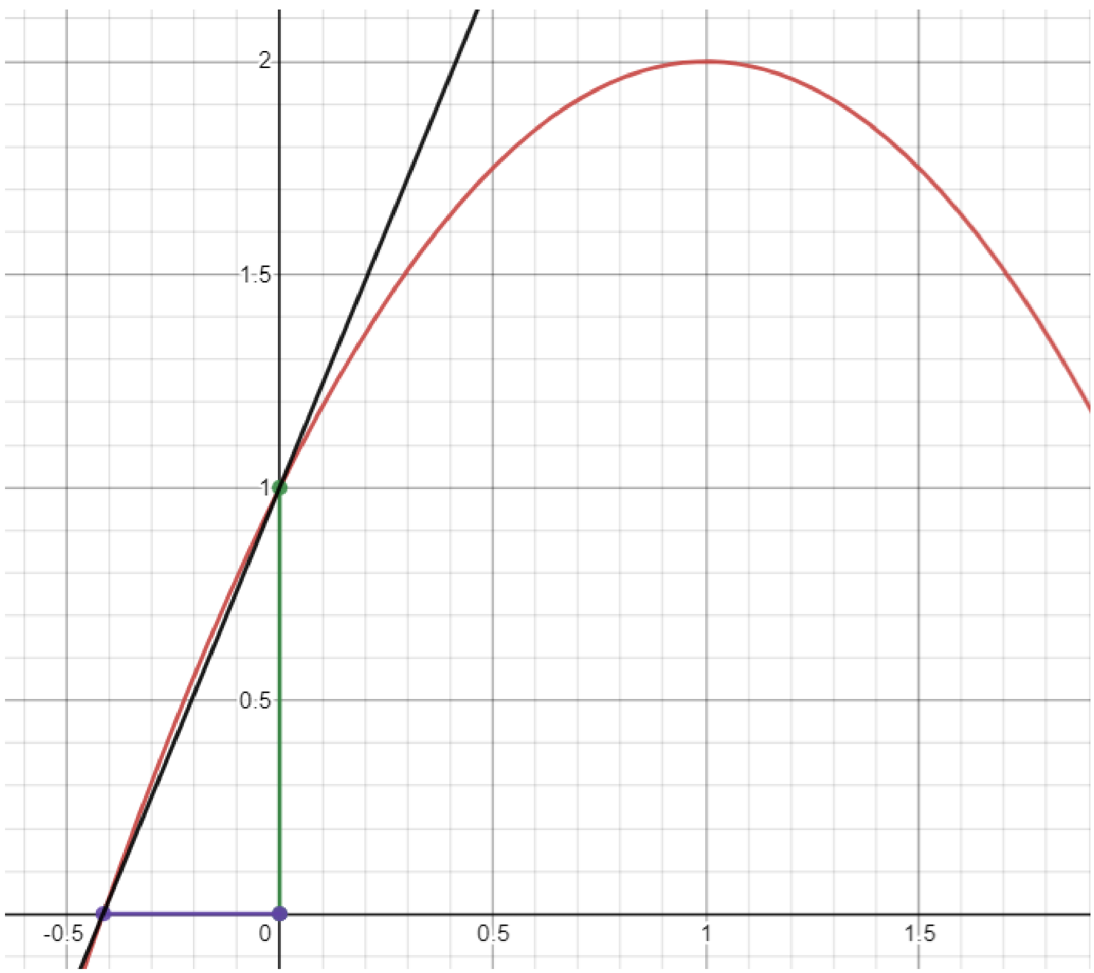


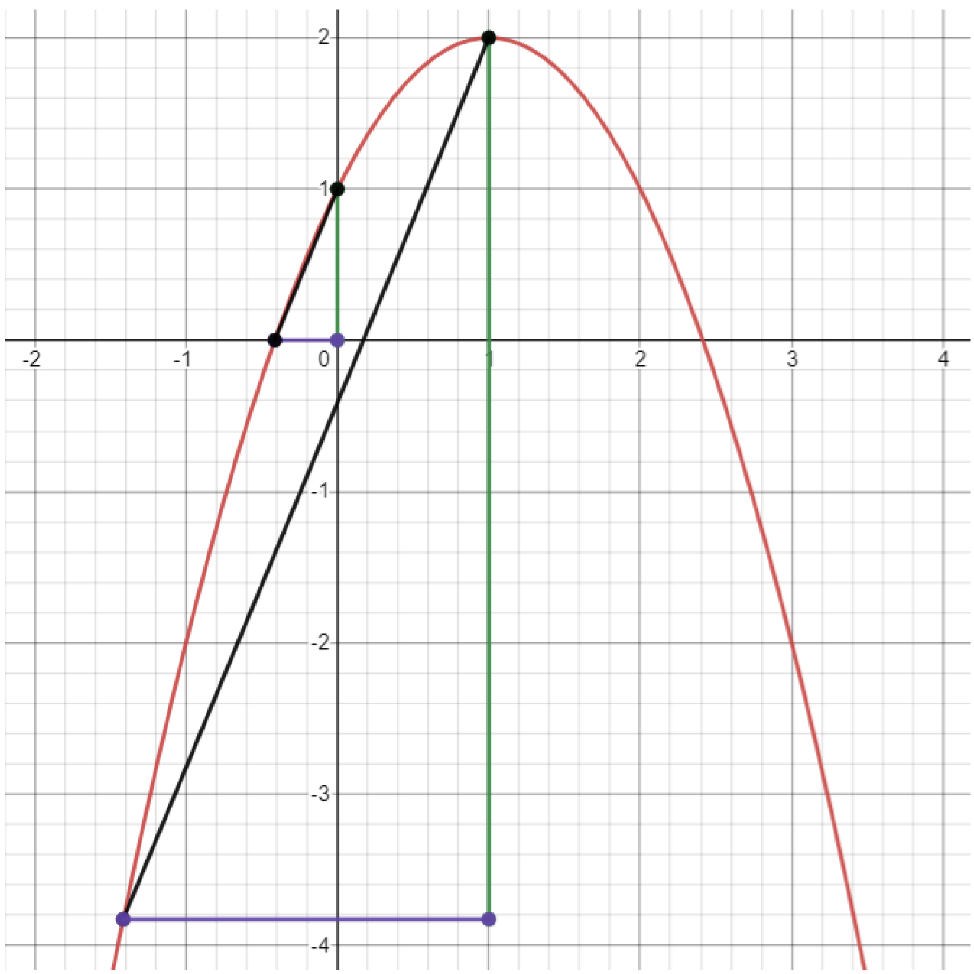
Image created by author using Desmos.

Define the slope at the point $(0, c)$.	$m_1 = b$
Define the slope at the point $(-k, 0)$.	$m_2 = \sqrt{b^2 - 4ac}$
Refer to the Secant-Slope Theorem.	$m_s = \dfrac{m_1 + m_2}{2}$
Apply the Secant-Slope Theorem to these two points.	$m_s = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
Refer to the definition of a slope and apply to c and k.	$\dfrac{\Delta y}{\Delta x} = \dfrac{c}{k}$
Equate with m_s.	$\dfrac{\Delta y}{\Delta x} = \dfrac{c}{k} = m_s = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
More simply.	$\dfrac{c}{k} = \dfrac{b + \sqrt{b^2 - 4ac}}{2}$
Solve for k.	$k = \dfrac{2c}{b + \sqrt{b^2 - 4ac}}$

Proof of K-Value by Similar Triangles

$$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

The value k can be solved using similar triangles. Consider a right triangle of the three points $(0,0)$, $(0, c)$, and $(-k, 0)$. A similar triangle can be constructed using the Tangent Point Y-Value Theorem, the Tangent Point X-Value Theorem, and the Secant-Vertex Theorem.



The large similar triangle is constructed using various theorems.

Image created by author using Desmos.

The triangle exists with the vertices $(0,0)$, $(-k,0)$ and $(0,c)$. To construct a similar triangle, the point corresponding to $(-k,0)$ must be found. Refer to the Parabola Point Equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a}+c\right)$
Let the point in question be named Point T and defined as having the tangent with the slope m_T. Then the point equation for Point T is defined by	$\left(\frac{m_T-b}{2a}, \frac{{m_T}^2-b^2}{4a}+c\right)$
As described in the previous proof, the hypotenuse connecting $(-k,0)$ and $(0,c)$ has the slope defined by	$m_s = \frac{b+\sqrt{b^2-4ac}}{2}$
Since we have similar triangles, the slope from the vertex to Point T must have the same slope, such that	$m_v = m_s = \frac{b+\sqrt{b^2-4ac}}{2}$
Refer to the Secant-Vertex Theorem.	$m_v = \frac{m_2}{2}$
Since the slope at the vertex is 0, the slope of the secant is half that of the tangent of Point T.	$m_v = \frac{m_T}{2} = \frac{b+\sqrt{b^2-4ac}}{2}$
This gives the tangent slope of Point T.	$m_T = b+\sqrt{b^2-4ac}$
Using the Tangent Point X-Value Theorem and the Tangent Point Y-Value Theorem, the lengths of the legs of the similar triangle are obtained.	$l_x = -\frac{m_T}{2a}$

	$l_y = -\frac{{m_T}^2}{4a}$
The ratio of the similar triangles is given by	$\frac{k}{c} = \frac{l_x}{l_y}$
More exactly.	$\frac{k}{c} = \frac{-\frac{m_T}{2a}}{-\frac{{m_T}^2}{4a}}$
Cross multiply.	$\frac{k}{c} = -\frac{m_T}{2a} \cdot -\frac{4a}{{m_T}^2}$
Simplify.	$\frac{k}{c} = \frac{2}{m_T}$
Solve for k.	$k = \frac{2c}{m_T}$
Insert the value for m_T.	$k = \frac{2c}{b + \sqrt{b^2 - 4ac}}$

With the value of k ascertained, the 2C Quadratic Formula is confirmed.

$$x_0 = 0 - \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

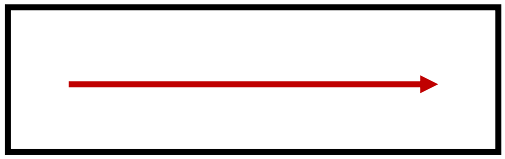
$$x_0 = -\frac{b}{a} + \frac{2c}{b + \sqrt{b^2 - 4ac}}$$

#10
Matching Method

I have tutored many Algebra students, who were tasked with the “guess and check” method of factoring quadratics. This typically occurs when $a \neq 1$. It can be quite a frustrating and onerous task, one that leaves the tables covered with the detritus of worn-down erasers. I developed this strategy as a more systematic approach to the “guess and check” method. This is best taught by example.

Developed July 2023.

MATCHING METHOD



Consider the parabola:

$$y = ax^2 + bx + c$$

We begin with an easy example:

$$y = 2x^2 + 11x + 12$$

The student will list the factors of a and c vertically in columns.

1 2		1 12
		2 6
		3 4

The student will multiply the left units by the right units in both possible ways, until the student finds two products whose sum is 11.

1 2		1 12
		2 6
		3 4

The factors are $(2x + 3)(x + 4) = 0$. Note that the a factors points to the c factors of the complementing, not shared, parenthetical factors. The student will find the roots by solving for x for each factor.

$$2x + 3 = 0, x = -\frac{3}{2}$$

$$x + 4 = 0, x = -4$$

A more difficult example:

$$y = 6x^2 + 17x + 12$$

The student will list the factors of a and c vertically in columns.

1 6		1 12
2 3		2 6
		3 4

The student will multiply the left units by the right units in both possible ways, until the student finds two products whose sum is 17.

1 6		1 12
2 3		2 6
		3 4

The factors are $(2x + 3)(3x + 4) = 0$.

An even more difficult example:

$$y = 2x^2 + 10x - 12$$

The student will list the factors of a and c vertically in columns.

1 2		-1 12	1 -12
		-2 6	2 -6
		-3 4	3 -4

The student will multiply the left units by the right units in both possible ways, until the student finds two products whose sum is 10.

1 2		-1 12	1 -12
		-2 6	2 -6
		-3 4	3 -4

The factors are $(2x - 2)(x + 6) = 0$.

Proofs of Supporting Theorems

This chapter is in five sections:

(1) The proof of the y_m shortcuts. These are relatively straightforward, using the definition of x_m that is well-known to every algebra teacher.

(2) A list of parabola "point equations" (as I prefer to call them). They are necessary for the theorems below.

(3) Four theorems necessary for the proofs of the Vertex Method (and its variants reliant on the relationship $\frac{L}{2} = \sqrt{|y_m|}$) and the 2C Method. These theorems were developed for my paper *Mathematica Exploratio I* and are taken directly from there.

(4) Two proofs establishing the relationship $|m| = L$.

(5) Two proofs establishing the relationship $\frac{L}{2} = \sqrt{|y_m|}$.

PROOFS OF y_m SHORTCUTS

SHORTCUT #1

Evidence of these relationships become apparent to any teacher who has taught Algebra II long enough. The proofs are helpful.

$$y_m = -a{x_m}^2 + c$$

	$y = ax^2 + bx + c$
Let $x = x_m = -\frac{b}{2a}$.	$y_m = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$
Apply and reduce.	$y_m = \frac{b^{\ 2}}{4a} - \frac{b^2}{2a} + c$
Let $Q = \frac{b^2}{2a}$.	$y_m = \frac{Q}{2} - Q + c$
Solve.	$y_m = -\frac{Q}{2} + c$
Replace $Q = \frac{b^2}{2a}$.	$y_m = -\frac{b^2}{4a} + c$
Generate ${x_m}^2$.	$y_m = \frac{b^2}{4a^2} \cdot K + c$
Solve for K to maintain equivalent.	$y_m = \frac{b^2}{4a^2} \cdot (-a) + c$
Shortcut 1 is proven.	$y_m = {x_m}^2 \cdot (-a) + c$

SHORTCUT #2

$$y_m = \frac{b}{2} x_m + c$$

	$y = ax^2 + bx + c$
Let $x = x_m = -\frac{b}{2a}$.	$y_m = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c$
Apply and reduce.	$y_m = \frac{b^2}{4a} - \frac{b^2}{2a} + c$
Let $Q = \frac{b^2}{2a}$.	$y_m = \frac{Q}{2} - Q + c$
Solve.	$y_m = -\frac{Q}{2} + c$
Replace $Q = \frac{b^2}{2a}$.	$y_m = -\frac{b^2}{4a} + c$
Extract $x_m = -\frac{b}{2a}$.	$y_m = -\frac{b}{2a} \cdot \frac{b}{2} + c$
Shortcut 2 is proven.	$y_m = x_m \cdot \frac{b}{2} + c$

PARABOLA POINT EQUATIONS

PARABOLA

$$P\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a}+c\right)$$

VERTEX

$$V\left(\frac{-b}{2a}, \frac{-b^2}{4a}+c\right)$$

FOCUS

$$F\left(\frac{-b}{2a}, \frac{1-b^2}{4a}+c\right)$$

DIRECTRIX POINT

$$D\left(\frac{-b}{2a}, \frac{-1-b^2}{4a}+c\right)$$

PARABOLA THEOREMS

SECANT-SLOPE THEOREM

$$m_s = \frac{m_1 + m_2}{2}$$

The slope m_s of a secant passing through two points of a parabola, where m_1 and m_2 are the slopes of the tangents at those points, is given by

$$m_s = \frac{m_1 + m_2}{2}$$

This holds whether the secant does or does not cross the axis of symmetry.

Using the parabola point equation, consider two points on a parabola distinguished by slopes m_1 and m_2.	$\left(\frac{m_1 - b}{2a}, \frac{{m_1}^2 - b^2}{4a} + c\right)$ $\left(\frac{m_2 - b}{2a}, \frac{{m_2}^2 - b^2}{4a} + c\right)$
Let the slope of the secant s be given by	$m_s = \frac{y_1 - y_2}{x_1 - x_2}$
Insert the x and y values from the parabola point equations into the equation for m_s.	$m_s = \frac{\left(\frac{{m_1}^2 - b^2}{4a} + c\right) - \left(\frac{{m_2}^2 - b^2}{4a} + c\right)}{\frac{m_1 - b}{2a} - \frac{m_2 - b}{2a}}$

Apply algebra.	$m_s = \dfrac{\dfrac{{m_1}^2 - {m_2}^2}{4a}}{\dfrac{m_1 - m_2}{2a}}$
Factor the upper numerator.	$m_s = \dfrac{\dfrac{(m_1 - m_2)(m_1 + m_2)}{4a}}{\dfrac{m_1 - m_2}{2a}}$
Multiply by the reciprocal of the denominator.	$m_s = \dfrac{m_1 + m_2}{2}$

SECANT-VERTEX THEOREM

$$m_v = \frac{m_2}{2}$$

The slope m_v of a secant passing through the vertex and any other point on the parabola, where m_1 and m_2 are the slopes of the tangents at those points, is given by

$$m_v = \frac{m_2}{2}$$

Refer to the Secant-Slope Theorem	$m_s = \frac{m_1 + m_2}{2}$
The slope of the tangent passing through the vertex equals 0.	$m_1 = 0$
Apply the m_1 value to the Secant-Slope Equation.	$m_s = \frac{0 + m_2}{2}$
The theorem is proven.	$m_v = \frac{m_2}{2}$

TANGENT POINT Y-VALUE THEOREM

$$l_y = -\frac{m^2}{4a}$$

Let m be the slope of the tangent of any point of a parabola except for the vertex. Let l_y represent the y-value distance between that point and the vertex. The length l_y will be given by the equation:

$$l_y = -\frac{m^2}{4a}$$

Refer to the vertex point equation.	$\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$
Refer to the parabola point equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a} + c\right)$
Let l_y equal the y-value of the vertex minus the y-value of the tangent point.	$l_y = \left(\frac{-b^2}{4a} + c\right) - \left(\frac{m^2-b^2}{4a} + c\right)$
Apply algebra.	$l_y = \left(\frac{-b^2}{4a}\right) - \left(\frac{m^2-b^2}{4a}\right)$
The theorem is proven.	$l_y = -\frac{m^2}{4a}$

TANGENT POINT X-VALUE THEOREM

$$l_x = -\frac{m}{2a}$$

Let m be the slope of the tangent of any point of a parabola except for the vertex. Let l_x represent the x-value distance between that point and the vertex. The length l_x will be given by the equation:

$$l_x = -\frac{m}{2a}$$

Refer to the vertex point equation.	$\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)$
Refer to the parabola point equation.	$\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a} + c\right)$
Let l_x equal the x-value of the vertex minus the x-value of the tangent point.	$l_x = \left(\frac{-b}{2a}\right) - \left(\frac{m-b}{2a}\right)$
The theorem is proven.	$l_x = -\frac{m}{2a}$

PROOFS OF SLOPE-LENGTH THEOREM

$$|m| = L$$

PROOF #1

Consider the variable L representing the distance between the roots of a parabola.	$L = x_{0,1} - x_{0,2}$
Refer to the Tangent Point X-Value Theorem.	$l_x = -\frac{m}{2a}$
Since L is defined by $L = 2l_x$, the equation for L becomes:	$L = 2 \cdot l_x = 2 \cdot -\frac{m}{2a} = -\frac{m}{a}$
The slope is positive or negative depending on which root is considered. Since that choice arbitrary, we have:	$L = \left\|-\frac{m}{a}\right\|$
For the monic parabola, this is expressed as:	$L = \|m\|$

PROOF #2

Consider the parabola point equation.	$P\left(\frac{m-b}{2a}, \frac{m^2-b^2}{4a}+c\right)$
Ascertain the y equation.	$y = \frac{m^2-b^2}{4a}+c$
Adjust the second term.	$y = \frac{m^2-b^2}{4a}+\frac{4ac}{4a}$
Combine.	$y = \frac{m^2-b^2+4ac}{4a}$
Multiply by $4a$.	$4ay = m^2 - b^2 + 4ac$
Re-arrange to solve for m.	$m^2 = b^2 + 4ay - 4ac$
Extract $4a$.	$m^2 = b^2 + 4a(y-c)$
Solve for m.	$m = \sqrt{b^2 + 4a(y-c)}$
Consider the value of m when $y = 0$.	$m = \sqrt{b^2 - 4ac}$
Consider the Quadratic Formula.	$x_0 = midpoint \pm \frac{L}{2}$
Therefore:	$L = \|m\|$

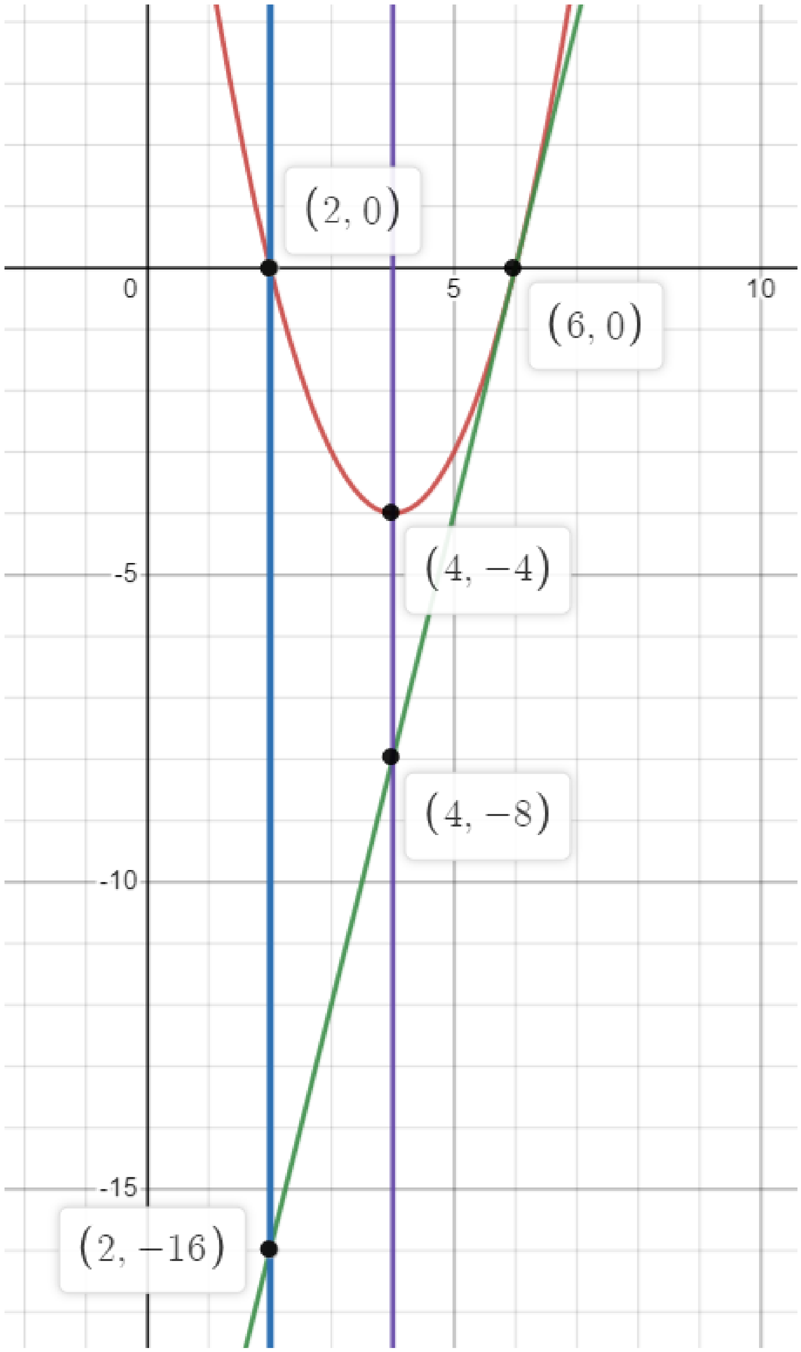
PROOFS OF HEIGHT-LENGTH THEOREM

$$\frac{L}{2} = \sqrt{y_m}$$

PROOF #1

Refer to the Tangent Point X-Value Theorem.	$l_x = -\frac{m}{2a}$
The length L between the two roots is twice this value; therefore:	$\left\|\frac{L}{2}\right\| = \|l_x\| = \left\|-\frac{m}{2a}\right\| = \frac{m}{2a}$
Refer to the Tangent Point Y-Value Theorem.	$l_y = -\frac{m^2}{4a}$
The roots occur at $y = 0$. From that reference point, l_y is equivalent to y_m.	$y_m = l_y = -\frac{m^2}{4a}$
The absolute value of the square root is expressed as:	$\sqrt{\|y_m\|} = \sqrt{\left\|-\frac{m^2}{4a}\right\|} = \frac{1}{\sqrt{a}} \cdot \frac{m}{2}$
Compare equations:	$\left\|\frac{L}{2}\right\| = \frac{m}{2a}$, $\sqrt{\|y_m\|} = \frac{m}{2\sqrt{a}}$
For the monic, $a = 1$, giving:	$\frac{L}{2} = -\frac{m}{2}$, $\sqrt{\|y_m\|} = \frac{m}{2}$
The proof is complete.	$\frac{L}{2} = \sqrt{\|y_m\|}$

PROOF #2



The slope of the tangent at point x_0 is equal to the distance between the roots.

Image created by author using Desmos.

Consider the parabola:

$$y = x^2 + bx + c$$

Refer to the Slope-Length Theorem.	$L = \|m\|$
Consider the slope (as a good high school student) as rise over run:	$m = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$
For $y = 0$, the run is equal to L.	$m = \frac{\Delta y}{L} = \frac{rise}{L}$
The slope has the same abosolute value as the distance between the roots. Therefore:	$L = \frac{rise}{L}$
Solve for the rise.	$L^2 = rise$
The value of the rise is four times the height of the parabola.	$rise = 4y_m$
Substitute.	$L^2 = 4y_m$
Solve for L.	$L = 2\sqrt{\|y_m\|}$
The proof is complete.	$\frac{L}{2} = \sqrt{\|y_m\|}$

Acknowledgements of Previous Studies

Although I developed these methods myself and they are "original to me," that does not mean that others have not arrived at these methods before I have. Therefore, it is incumbent upon me to do what research that I can to see if anyone has. To date, I can say that I have not found these methods anywhere, except for an expression resembling the monic form of the X-M Method. But as an amateur, I do not think I can do a superlative job of this research, simply because I do not have access to the same literature as a professional mathematician, so I apologize in advance for my deficiencies. What I have found, I share with you here.

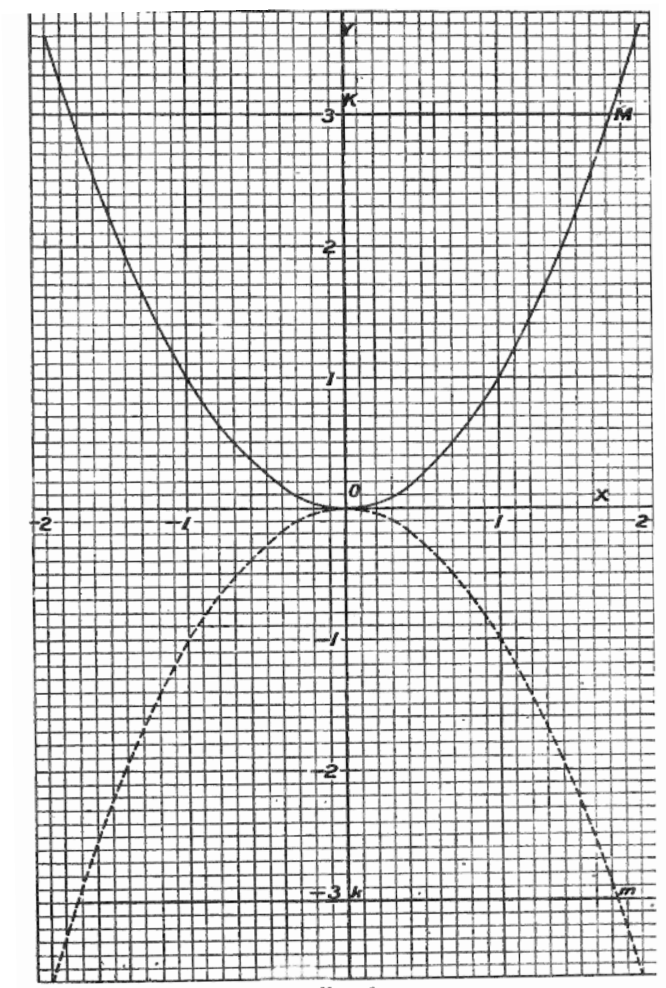


Fig. 1.

In 1904, the mathematicians J.G. Hamilton and F. Kettle published *Graphs and Imaginaries* (Edward Arnold Publishers Ltd.), which includes two chapters on solving quadratic equations. The book includes the image, which I include here (as it is no longer restricted by copyright).

The image displays two parabolas, the parabola on the real plane (solid line) and the "shadow" parabola on the imaginary plane, $y = x^2$and $y = -x^2$, respectively. The authors state that "since the abscissa of any point on the curve is the square root of the ordinate of that point, it is immediately obvious from their construction that

$$\sqrt{OK} = \pm KM$$

and

$$\sqrt{-OK} = \pm i.KM$$

kO being equal to OK and km to KM" (p. 10). This assertion establishes the relationship between the x and y values of the $y = x^2$and $y = -x^2$ parabolas by the expressions of the equations themselves. But the concept is notable: The square root of a length in the y-direction is equal to the corresponding length in the x-direction. The concept can be applied to all parabolas, though it needs to be adjusted when $a \neq 1$. The Vertex Method, the Circular Paraboloid Method and the Big Sister Method rely on this basic concept.

In Ron Irving's book *Beyond the Quadratic Formula* (MAA Press, 2013, p.26), the author considers the monic quadratic

$$x^2 + bx + c = 0$$

He substitutes variables, such that $B = \frac{b}{2}$ and $C = -c$. He arrives at the form:

$$x = -B \pm \sqrt{B^2 + C}$$

This expression of the Quadratic Formula is very close to the monic form of the X-M Method, which would be expressed as

$$x_0 = x_m \pm \sqrt{x_m^2 - c}$$

Irving's excellent book includes a history of the Quadratic Formula and an in-depth investigation of Cardano's Formula.

Po-Shen Loh introduced a novel method in 2019. He begins with $-\frac{b}{2}$ as the average of the two roots and introduces the variable z as the unknown difference from the average, such that

$$\left(-\frac{b}{2} + z\right)\left(-\frac{b}{2} - z\right) = c$$

$$\left(-\frac{b}{2}\right)^2 - z^2 = c$$

Solving for z, the students adds that value to the average, obtaining

$$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

This is a reduced monic form of the Quadratic Formula.

Typical high school texts teach the Quadratic Formula algebraically, deriving it by "completing the square" (e.g. Harshbarger, 1976; Baratto & Bergman, 2008). This is useful since completing the square has other applications that students will encounter in future studies. However, since it is completely algebraic, it lacks helpful visuals relating the Quadratic Formula to parabolas. The methods described throughout this short book provide opportunities for students to visualize parabolas by applying alternate forms of the Quadratic Formula in a variety of fascinating and complementary ways.

REFERENCES

Baratto, S., & Bergman, B. [2008.] *Beginning Algebra*. Vol 2. 7th ed. McGraw Hill.

Hamilton, J.G. & Kettle, F. [1904.] *Graphs and Imaginaries: An Easy Method of Finding Graphically Imaginary Roots of Quadratic Equations and Imaginary Points of Intersections of Various Curves with Illustrations of the Principle from Elementary Geometry*. London: Edward Arnold Publishing Ltd. Reprinted by HardPress Publishing.

Harshbarger, R. [1976.] *Intermediate Algebra*. NY: Harper & Row.

Herrmann, F. [2021, June]. *Mathematica Exploratio I*.

Herrmann, F. [2021, September]. *The 3-Space Method for Solving Parabolic Roots*.

Irving, R. [2013.] *Beyond the Quadratic Formula*. Providence: MAA Press.

Loh, P. [2019, December.] *A Simple Proof of the Quadratic Formula*. arXiv:1910.06709v2

Images created using GeoGebra published with permission.

Images created with Desmos published with permission.